THE DUAL POLYHEDRAL PRODUCT, COCATEGORY AND NILPOTENCE

STEPHEN THERIAULT

Abstract. The notion of a dual polyhedral product is introduced as a generalization of Hovey's definition of Lusternik-Schnirelmann cocategory. Properties established from homotopy decompositions that relate the based loops on a polyhedral product to the based loops on its dual are used to show that if X is a simply-connected space then the weak cocategory of X equals the homotopy nilpotency class of ΩX. This answers a fifty year old problem posed by Ganea. The methods are applied to determine the homotopy nilpotency class of quasi-p-regular exceptional Lie groups and sporadic p-compact groups.

1. Introduction

This paper establishes strong relationships between three different concepts in topology: polyhedral products, cocategory and homotopy nilpotency. Polyhedral products play a fundamental role in toric topology and have a growing number of applications to other areas of mathematics, such as group actions on graphs, intersections of quadrics, and coordinate subspace arrangements. Cocategory is dual to Lusternik-Schnirelmann category; the latter has been heavily studied since it was introduced in the 1930s, partly because of its connection to counting the critical points of functions between smooth manifolds. Homotopy nilpotency is the topological analogue of nilpotency in group theory, and it has powerful implications for homotopy theory, especially in the stable case.

Throughout the paper, assume that all spaces have the homotopy type of path-connected CW-complexes. It has long been thought that cocategory and homotopy nilpotency are very closely linked. One problem in establishing a good link is settling on the right definition of cocategory. Several different definitions exist, each trying to dualize some feature of Lusternik-Schirelmann category. We use Hovey's definition, but others include those by Ganea [G1], Hopkins [Hop] and Murillo-Viruel [MV]. In all cases, a connection is made between the definition of cocategory in question and homotopy nilpotence. For example, Hovey and Murillo-Viruel show that, in their own notion of cocategory, if the cocategory of a simply-connected space X is m then iterated Samelson products of length $\geq m+1$ formed from the homotopy groups of ΩX all vanish. Our main result is to show that if X is a simply-connected space then the weak cocategory of X (using Hovey's definition) is precisely equal to the homotopy nilpotency class of ΩX.

2010 *Mathematics Subject Classification.* Primary 55M30, 55P15, 55P35.

Key words and phrases. polyhedral product, thin product, cocategory, Whitehead product, homotopy nilpotence.

To do this we first generalize the notion of cocategory to dual polyhedral products, and then use loop space decompositions to compare the based loops on a polyhedral product to the based loops on its dual. This results in a natural filtration of the based loops on a polyhedral product that should have many applications beyond those in this paper. A more detailed description is obtained in the special case of a thin product, which is relevant to cocategory.

The methods developed in the paper are sufficiently powerful to allow for explicit calculations of the homotopy nilpotency classes of quasi-p-regular exceptional Lie groups and nonmodular p-compact groups. These cases would have previously been considered as completely unapproachable.

To describe our results more carefully, several definitions are required.

The dual polyhedral product. Let K be a simplicial complex on m vertices. For $1 \leq i \leq m$, let (X_i, A_i) be a pair of pointed CW-complexes, where A_i is a pointed subspace of X_i. Let $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^m$ be the set of CW-pairs. For each simplex (face) $\sigma \in K$, let $(\underline{X}, \underline{A})^\sigma$ be the subspace of $\prod_{i=1}^m X_i$ defined by

$$(\underline{X}, \underline{A})^\sigma = \prod_{i=1}^m Y_i \qquad \text{where} \qquad Y_i = \begin{cases} X_i & \text{if } i \in \sigma \\ A_i & \text{if } i \notin \sigma. \end{cases}$$

The *polyhedral product* determined by $(\underline{X}, \underline{A})$ and K is

$$(\underline{X}, \underline{A})^K = \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma \subseteq \prod_{i=1}^m X_i.$$

For example, suppose each A_i is a point. If K is a disjoint union of m points then $(\underline{X}, \underline{*})^K$ is the wedge $X_1 \vee \cdots \vee X_m$, and if K is the standard $(m-1)$-simplex then $(\underline{X}, \underline{*})^K$ is the product $X_1 \times \cdots \times X_m$.

The polyhedral product is therefore a colimit over all the faces of K, ordered by inclusion. There is another way to describe the polyhedral product as a colimit. Let $[m] = \{1, \ldots, m\}$ and let $I = \{i_1, \ldots, i_k\}$ be a subset of $[m]$. Let K_I be the full subcomplex of K on the vertex set I, that is, the faces of K_I are those faces of K whose vertices are all in I. There is a map of simplicial complexes $K_I \longrightarrow K$ including K_I into K, and if the subsets of $[m]$ are ordered by inclusion then there is a system of simplicial maps $K_I \longrightarrow K_J$ whenever $I \subseteq J$. These induce maps of polyhedral products $\iota_{I,J} \colon (\underline{X}, \underline{A})^{K_I} \longrightarrow (\underline{X}, \underline{A})^{K_J}$. Notice that the only possible face of K which is not in some K_I for a proper subset I of $[m]$ is $\sigma = [m]$. But in this case K equals the full simplex Δ^{m-1}, and the polyhedral product $(\underline{X}, \underline{A})^K$ equals $X_1 \times \cdots \times X_m$. Therefore, if $K \neq \Delta^{m-1}$, then

$$(\underline{X}, \underline{A})^K = \bigcup_{I \subsetneq [m]} (\underline{X}, \underline{A})^{K_I}.$$

The description of the polyhedral product as a colimit of its full subcomplexes lets us define a dual notion. Fix a simplicial complex K on the vertex set $[m]$. Since each $(\underline{X}, \underline{A})^{K_I}$ is a pointwise inclusion into $(\underline{X}, \underline{A})^K$, it is a cofibration but not a fibration so the appropriate dual notion for

homotopy theory is not an inverse limit but a homotopy inverse limit. Order the subsets of $[m]$ by reverse inclusion. If $I \subseteq J$ then there does not exist a simplicial map $K_J \longrightarrow K_I$. However, there is a map on the level of polyhedral products. If X^I is the product $X_{i_1} \times \cdots \times X_{i_k}$ then, as in [DS, 2.2.3], the projection $X^J \longrightarrow X^I$ induces a map of polyhedral products $\varphi_{J,I} \colon (\underline{X}, \underline{A})^{K_J} \longrightarrow (\underline{X}, \underline{A})^{K_I}$ with the property that $\varphi_{J,I} \circ \iota_{I,J}$ is the identity map on $(\underline{X}, \underline{A})^{K_I}$. Assemble $\{(\underline{X}, \underline{A})^{K_I} \mid I \subseteq [m]\}$ and $\{\varphi_{J,I} \mid I \subseteq J \subseteq [m]\}$ as the vertices and edges of an m-cube in order to take a homotopy inverse limit.

Definition 1.1. If $K \neq \Delta^{m-1}$ then the *dual polyhedral product* is

$$(\underline{X}, \underline{A})^K_D = \underset{\substack{\longleftarrow \\ I \subsetneq [m]}}{\mathrm{holim}}\, (\underline{X}, \underline{A})^{K_I}.$$

Cocategory. A special case of the dual polyhedral product is the thin product. Let K be the simplicial complex on the vertex set $[m]$ consisting of m disjoint points. In each pair of spaces (X_i, A_i), take $A_i = *$. Then for $I \subseteq [m]$ the full subcomplex K_I consists of $|I|$ disjoint points, so

$$(\underline{X}, \underline{*})^{K_I} = \bigvee_{i \in I} X_i.$$

If $J \subseteq I$, the projection $K_I \longrightarrow K_J$ induces a map of polyhedral products $\bigvee_{i \in I} X_i \longrightarrow \bigvee_{j \in J} X_j$ which sends X_i to itself if $i \in J$ or to the basepoint if $i \notin J$. Write $\underline{X}$ for the set of spaces $\{X_1, \cdots, X_m\}$.

Definition 1.2. The *thin product* of pointed spaces $X_1, \ldots X_m$ is the space

$$P^m(\underline{X}) = (\underline{X}, \underline{*})^K_D$$

where K is the simplicial complex consisting of m disjoint points.

The thin product was defined by Hovey [Hov] as a dual to the fat wedge, and he used it to define a notion of cocategory that is dual to Lusternik-Schnirelmann category. Some properties of the thin product have been determined by Hovey, Anick [A] in the case when $m = 3$, and Chorney and Scherer [CS] in the context of Goodwillie towers. Other properties more directed towards cocategory have been determined by Sbaï [Sb] in the rational case and by Krafi and Mamouni [KM] in the context of automorphism groups.

In the case when each X_i equals a common space X for $1 \leq i \leq m$, write $P^m(X)$ for $P^m(\underline{X})$. Let

$$\nabla \colon \bigvee_{i=1}^{m} X \longrightarrow X$$

be the m-fold folding map.

Definition 1.3. Let X be a pointed space. The *cocategory* of X is the least m for which there exists an extension

$$\begin{tikzcd} \bigvee_{i=1}^{m+1} X \arrow[r] \arrow[d, "\nabla"] & P^{m+1}(X) \arrow[dl, dashed] \\ X. & \end{tikzcd}$$

In this case, write $\mathrm{cocat}(X) = m$.

For example, when $m = 1$ then $P^2(X)$ is the homotopy inverse limit of the system $X \longrightarrow * \longleftarrow X$. That is, $P^2(X) \simeq X \times X$. So $\mathrm{cocat}(X) = 1$ if and only if X is an H-space.

There is a weaker version of cocategory which will be important. Define the space $F^{m+1}(X)$ and the map $f^{m+1}(X)$ by the homotopy fibration

$$F^{m+1}(X) \xrightarrow{f^{m+1}(X)} \bigvee_{i=1}^{m+1} X \longrightarrow P^{m+1}(X).$$

Definition 1.4. Let X be a pointed space. The *weak cocategory* of X is the least m for which the composite

$$\begin{tikzcd} F^{m+1}(X) \arrow[r, "f^{m+1}(X)"] & \bigvee_{i=1}^{m+1} X \arrow[d, "\nabla"] \\ & X \end{tikzcd}$$

is null homotopic. In this case, write $\mathrm{wcocat}(X) = m$.

Notice that the definitions immediately imply that $\mathrm{wcocat}(X) \leq \mathrm{cocat}(X)$.

Homotopy nilpotency. An *H-group* is a homotopy associative H-space with a homotopy inverse. Let G be an H-group. The commutator $\bar{c}\colon G \times G \longrightarrow G$ is defined pointwise by $\bar{c}(x, y) = xyx^{-1}y^{-1}$. Observe that the restriction of $\bar{c}$ to the wedge is null homotopic so $\bar{c}$ extends to a map

$$c\colon G \wedge G \longrightarrow G.$$

Since $\Sigma G \wedge G$ is a retract of $\Sigma(G \times G)$, the homotopy class of c is uniquely determined by that of $\bar{c}$. The map c is the Samelson product of the identity map on G with itself. For an integer $m \geq 1$, let $G^{(m+1)}$ be the $(m + 1)$-fold smash product of G with itself. Define the iterated Samelson product

$$c_m\colon G^{(m+1)} \longrightarrow G$$

by $c_m = c \circ (1 \wedge c) \circ \cdots \circ (1 \wedge \cdots 1 \wedge c)$. Notice that c_m has a universal property: any Samelson product of length $m + 1$ on G factors through c_m.

Definition 1.5. Let G be an H-group. The *homotopy nilpotency class* of G is the least m such that c_m is null homotopic but c_{m-1} is not. In this case, write $\mathrm{nil}(G) = m$.

For example, $\mathrm{nil}(G)=1$ if and only if G is homotopy commutative.

Our main theorem identifies weak cocategory and homotopy nilpotency.

Theorem 1.6. *Let X be a simply-connected space. Then* $\mathrm{wcocat}(X)=m$ *if and only if* $\mathrm{nil}(\Omega X)=m$.

The approach to proving Theorem 1.6 is to consider the homotopy fibration $F^m(\underline{X})\xrightarrow{f^m(\underline{X})}\bigvee_{i=1}^m X_i\longrightarrow P^m(\underline{X})$. We identify the homotopy type of $F^m(\underline{X})$ as a certain wedge of suspensions, and the homotopy class of $f^m(\underline{X})$ as a wedge sum of Whitehead products. In the case when each X_i equals a common space X, this lets us play off the definition of weak cocategory, which involves the map $f^m(X)$, with the homotopy nilpotentcy of ΩX by taking the adjoints of the Whitehead products to obtain Samelson products.

The identifications for $F^m(\underline{X})$ and $f^m(\underline{X})$ are obtained as special cases of much more general phenomena involving dual polyhedral products. The definition of $(\underline{X},\underline{A})^K_D$ as a homotopy inverse limit implies that there is a map $(\underline{X},\underline{A})^K\longrightarrow(\underline{X},\underline{A})^K_D$. Let $F_{[m]}$ be its homotopy fibre. By comparing the homotopy types of $\Omega(\underline{X},\underline{A})^K$ and $\Omega(\underline{X},\underline{A})^K_D$, we show that there is a homotopy equivalence

$$\Omega(\underline{X},\underline{A})^K\simeq\Omega(\underline{X},\underline{A})^K_D\times\Omega F_{[m]} \tag{1}$$

In a way that can be made precise via certain idempotents, $\Omega(\underline{X},\underline{A})^K_D$ contains all the information about $\Omega(\underline{X},\underline{A})^K$ that involves only proper subsets of the ingredient pairs (X_i,A_i), while $\Omega F_{[m]}$ captures all of the information about $\Omega(\underline{X},\underline{A})^K$ that involves all m pairs (X_i,A_i) simultaneously. This leads to a filtration of the homotopy theory of $\Omega(\underline{X},\underline{A})^K$ obtained from its full subcomplexes.

Theorem 1.7. *For any polyhedral product $(\underline{X},\underline{A})^K$, there is a homotopy equivalence*

$$\Omega(\underline{X},\underline{A})^K\simeq\prod_{I\subsetneq[m]}\Omega F_I$$

where F_I is the homotopy fibre of the map $(\underline{X},\underline{A})^{K_I}\longrightarrow(\underline{X},\underline{A})^{K_I}_D$.

In the special case when $(\underline{X},\underline{A})$ is of the form $(\underline{CX},\underline{X})$, where CX is the reduced cone on X, more can be said. The polyhedral product $(\underline{CX},\underline{X})^K$ has been well studied. In [BBCG] it is shown that there is a homotopy equivalence

$$\Sigma(\underline{CX},\underline{X})^K\simeq\bigvee_{I\notin K}\Sigma^2(|K_I|\wedge\widehat{X}^I) \tag{2}$$

where $|K_I|$ is the geometric realization of K_I, and for $I=\{i_1,\ldots,i_k\}$, we have $\widehat{X}^I=X_{i_1}\wedge\cdots\wedge X_{i_k}$. In particular, $\Sigma(\underline{CX},\underline{X})^K$ is a wedge of suspensions of iterated smashes of the spaces X_i.

A great deal of work has been done in [BG, GT2, GT3, IK1, IK2] to determine for which simplicial complexes the decomposition (2) desuspends. In [IK2] the notion of a totally homology fillable simplicial complex was introduced, which includes the more well known families of shifted, shellable

and sequentially Cohen-Macaulay complexes. It was shown that if K is totally homology fillable then there is a homotopy equivalence

$$(\underline{CX}, \underline{X})^K \simeq \bigvee_{I \notin K} \Sigma(|K_I| \wedge \widehat{X}^I) \tag{3}$$

and $\Sigma|K_I|$ is homotopy equivalent to a wedge of spheres. Consequently, $(\underline{CX}, \underline{X})^K$ is homotopy equivalent to a wedge of suspensions of iterated smashes of the spaces X_i.

When K is totally homology fillable we show that the spaces F_I in Theorem 1.7 are also homotopy equivalent to wedges of suspensions of iterated smashes of the X_i's. Further, in the special case of the thin product, the spaces $F_{[m]}$ and $F^m(\underline{X})$ are homotopy equivalent, and the map $F^m(\underline{X}) \xrightarrow{f^m(\underline{X})} \bigvee_{i=1}^m X_i$ is a wedge sum of iterated Whitehead products.

This paper is organized as follows. In Part I we give homotopy decompositions of $\Omega(\underline{X}, \underline{A})^K$ and $\Omega(\underline{X}, \underline{A})^K_D$ via certain idempotents. Section 2 establishes the basic decomposition in terms of certain telescopes and Section 3 identifies the telescopes as certain loop spaces. Section 4 refines the decomposition in the case of $\Omega(\underline{CX}, \underline{X})^K$ and $\Omega(\underline{CX}, \underline{X})^K_D$ when K is totally homology fillable by showing that the factors are the based loops on certain wedges of suspensions. In Section 5 these results are then transferred to give analogous decompositions in the case of $\Omega(\underline{X}, \underline{*})^K$ and $\Omega(\underline{X}, \underline{*})^K$ for the same complexes K.

In Part II the role of Whitehead products is investigated. This starts with fundamental results which seem to have no proofs in the literature. in Section 6 we show that Porter's decomposition of the homotopy fibre of the inclusion of a wedge into a product can be altered by a homotopy equivalence so that the maps from the fibre into the total space are described by Whitehead products. In Section 7 an anlagous result is shown for the homotopy fibre of the pinch map from a wedge onto one summand, and in Section 8 finer versions are proved in the case when each space in the wedge is a suspension. Finally, in Section 9 all this is applied to give a homotopy decomposition of the homotopy fibre of the map from the wedge into the thin product, and to identify the maps from the fibre to the wedge as certain Whitehead products.

In Part III the results from Parts I and II are used to prove Theorem 1.6 in Section 10. To explicitly calculate the homotopy nilpotency classes, a special class of H-spaces called retractile H-spaces is discussed in Section 11, and a criterion for applying Theorem 1.6 to retractile H-spaces is proved in Section 12. Examples are then given in Section 13.

Part 1. Homotopy decompositions of $\Omega(\underline{X}, \underline{A})^K$ and $\Omega(\underline{X}, \underline{A})^K_D$.

2. Decompositions via idempotents

The focus for the most part is on $\Omega(\underline{X}, \underline{A})^K$, with the decomposition for $\Omega(\underline{X}, \underline{A})^K_D$ being a consequence. The decomposition will be constructed by using a family of commuting idempotents.

The idempotents will be defined on $(\underline{X},\underline{A})^K$ but in order to take a product of their telescopes a multiplication is needed, which is why loop spaces are taken. We begin with some general information about decompositions of H-groups using idempotents.

In general, let G be a path-connected H-group. Suppose that for $1 \leq j \leq k$ there is a family of maps $e_j \colon G \longrightarrow G$ such that: (i) each e_j is an idempotent, (ii) $e_i \circ e_j \simeq *$ if $i \neq j$, and (iii) $1 \simeq e_1 + \cdots + e_k$, where 1 is the identity map on G and the addition refers to the group structure on $[G, G]$ induced by the multiplication on G. The family $\{e_j\}_{j=1}^k$ is called a set of *mutually orthogonal idempotents*. Let $T(e_j)$ be the telescope of e_j, that is, $T(e_j) = \underrightarrow{\mathrm{hocolim}}_{e_j} X$, and let $G \longrightarrow T(e_j)$ be the map to the telescope. Observe that $H_*(T(e_j)) \cong \mathrm{Im}\,(e_j)_*$. Since $e_i \circ e_j \simeq *$ for $i \neq j$ we may form the direct sum $\oplus_{j=1}^k H_*(T(e_j))$, and since $1 \simeq e_1 + \cdots + e_k$ we obtain an isomorphism of modules

$$H_*(G) \cong \oplus_{j=1}^k H_*(T(e_j)).$$

The product of the telescope maps

$$G \longrightarrow \prod_{j=1}^k T(e_j)$$

therefore induces an isomorphism in homology and so is a homotopy equivalence by Whitehead's Theorem. This is recorded for later use.

Lemma 2.1. *Let G be a path-connected H-group and suppose that $\{e_j\}_{j=1}^k$ is a family of mutually orthogonal idempotents on G. Then there is a homotopy equivalence $G \longrightarrow \prod_{j=1}^k T(e_j)$.* $\Box$

Next, suppose that for $1 \leq j \leq m$ there is a family of commuting idempotents $e_j \colon G \longrightarrow G$. Observe that each pair $(e_j, 1 - e_j)$ is mutually orthogonal, and the larger family of idempotents $\{e_j, 1 - e_j\}_{j=1}^m$ all commute. Let $\mathcal{J}$ be the collection of 2^m sequences $(a_1, \ldots, a_m)$, where each $a_j \in \{0, 1\}$. For $(a_1, \ldots, a_m) \in \mathcal{J}$, define

$$f_{(a_1,\ldots,a_m)} \colon X \longrightarrow X$$

by the composite

$$f_{(a_1,\ldots,a_m)} = f_{a_1} \circ \ldots \circ f_{a_m} \qquad \text{where} \qquad f_{a_j} = \begin{cases} e_j & \text{if } a_j = 0 \\ 1 - e_j & \text{if } a_j = 1. \end{cases}$$

We record three properties of the maps $f(a_1, \ldots, a_m)$. First, since the idempotents $\{e_j, 1 - e_j\}_{j=1}^m$ commute, each map $f_{(a_1,\ldots,a_m)}$ is also an idempotent. Second, if $f(a'_1, \ldots, a'_m)$ is another such idempotent distinct from $f(a_1, \ldots, a_m)$, then at least one $1 \leq j \leq m$ satisfies $a'_j \neq a_j$, for otherwise the two maps agree on every a_j and so are identical. Therefore the j^{th} term in the composite for $f(a_1, \ldots, a_m)$ is e_j and that for $f(a'_1, \ldots, a'_m)$ is $1 - e_j$, or vice-versa. But as $e_j \circ (1 - e_j)$ is null homotopic and the idempotents commute, we obtain $f(a_1, \ldots, a_m) \circ f(a'_1, \ldots, a'_m) \simeq *$. Third, since $1 = e_j + (1 - e_j)$ for each j, we obtain $1 = (e_1 + (1 - e_1)) \circ \cdots \circ (e_m + (1 - e_m))$.

Expanding gives $1 = \Sigma_{(a_1,\ldots,a_m)\in\mathcal{J}} f_{(a_1,\ldots,a_m)}$. The three properties together imply that the collection of idempotents $\{f_{(a_1,\ldots,a_m)}\}_{(a_1,\ldots,a_m)\in\mathcal{J}}$ is mutually orthogonal. Therefore, if $T(a_1,\ldots,a_m)$ is the telescope of $f_{(a_1,\ldots,a_m)}$ then Lemma 2.1 implies the following.

Lemma 2.2. *Let G be a path-connected H-group and suppose that $\{e_j\}_{j=1}^m$ is a family of commuting idempotents on G. Then there is a homotopy equivalence $G \longrightarrow \prod_{(a_1,\ldots,a_m)\in\mathcal{J}} T(a_1,\ldots,a_m)$.* □

We wish to construct a family of commuting idempotents on $(\underline{X},\underline{A})^K$. Recall from the Introduction that if $I = (i_1,\ldots,i_k) \subseteq [m]$ then K_I is a full subcomplex of K and there is a map of simplicial complexes $K_I \longrightarrow K$ but not a map $K \longrightarrow K_I$. However, the situation improves on the level of polyhedral products. Let $X^I = X_{i_1} \times \cdots \times X_{i_k}$, let $X^I \longrightarrow X^m$ be the map defined by sending the j^{th}-factor of X^I to the $(i_j)^{th}$-factor of X^m, and let $X^m \longrightarrow X^I$ be the projection. The inclusion of K_I into K induces a map $\iota_I \colon (\underline{X},\underline{A})^{K_I} \longrightarrow (\underline{X},\underline{A})^K$. The following lemma appears in [DS] and is an immediate consequence of the definition of the polyhedral product as union of coordinate subspaces of $X^m = X_1 \times \cdots \times X_m$.

Lemma 2.3. *Let K_I be a full subcomplex of K. The following hold:*

(a) *the inclusion $X^I \longrightarrow X^m$ induces a map of polyhedral products $(\underline{X},\underline{A})^{K_I} \longrightarrow (\underline{X},\underline{A})^K$ which equals ι_I;*
(b) *the projection $X^m \longrightarrow X^I$ induces a map of polyhedral products $\varphi_I \colon (\underline{X},\underline{A})^K \longrightarrow (\underline{X},\underline{A})^{K_I}$;*
(c) *the composite $(\underline{X},\underline{A})^{K_I} \xrightarrow{\iota_I} (\underline{X},\underline{A})^K \xrightarrow{\varphi_I} (\underline{X},\underline{A})^{K_I}$ is the identity map.*

□

For $1 \leq j \leq m$, let $I_j = [m]\backslash\{j\}$. By Lemma 2.3 there is a composite of polyhedral products

$$e_j \colon (\underline{X},\underline{A})^K \xrightarrow{\varphi_{I_j}} (\underline{X},\underline{A})^{K_{I_j}} \xrightarrow{\iota_{I_j}} (\underline{X},\underline{A})^K.$$

Lemma 2.4. *The following hold:*

(a) *for $1 \leq j \leq m$ the map e_j is an idempotent;*
(b) *for any $1 \leq j,k \leq m$ we have $e_j \circ e_k = e_k \circ e_j$.*

Proof. Part (a) follows immediately from the definition of e_j and Lemma 2.3 (c). For part (b), if $j = k$ then the statement is a tautology. Suppose that $j \neq k$. Let $I_{j,k} = [m]\backslash\{j,k\}$. Observe that the composites of projection and inclusion maps $X^m \longrightarrow X^{I_j} \longrightarrow X^m \longrightarrow X_{I_k} \longrightarrow X^m$ and $X^m \longrightarrow X^{I_k} \longrightarrow X^m \longrightarrow X_{I_j} \longrightarrow X^m$ both equal the composite $X^m \longrightarrow X^{I_{j,k}} \longrightarrow X^m$. Therefore, as in Lemma 2.3, $e_j \circ e_k = e_k \circ e_j$. □

By Lemma 2.4 the maps $\{e_j\}_{j=1}^m$ are commuting idempotents on $(\underline{X},\underline{A})^K$. However, this space is not an H-space in general so we must loop to obtain one. Since the loop map of an idempotent is

an idempotent, $\{\Omega e_j\}_{j=1}^m$ are commuting idempotents on $\Omega(\underline{X},\underline{A})^K$. For $(a_1,\ldots,a_m)\in\mathcal{J}$, define

$$f_{(a_1,\ldots,a_m)}\colon \Omega(\underline{X},\underline{A})^K \longrightarrow \Omega(\underline{X},\underline{A})^K$$

by the composite

$$f_{(a_1,\ldots,a_m)} = f_{a_1}\circ\ldots\circ f_{a_m} \qquad \text{where} \qquad f_{a_j} = \begin{cases} \Omega e_j & \text{if } a_j = 0 \\ 1-\Omega e_j & \text{if } a_j = 1. \end{cases}$$

Then each f_{a_j} is an idempotent and as $\{\Omega e_j\}_{j=1}^m$ commute, the composite $f_{(a_1,\ldots,a_m)}$ is also an idempotent. Let

$$T(a_1,\ldots,a_m) = \underrightarrow{\text{hocolim}}_{f_{(a_1,\ldots,a_m)}}\Omega(\underline{X},\underline{A})^K.$$

Lemma 2.2 implies the following.

Proposition 2.5. *Assume that $(\underline{X},\underline{A})^K$ is simply-connected. Then there is a homotopy equivalence*

$$\Omega(\underline{X},\underline{A})^K \simeq \prod_{(a_1,\ldots,a_m)\in\mathcal{J}} T(a_1,\ldots,a_m).$$

$\square$

We wish to relate the factors $T(a_1,\ldots,a_m)$ in the decomposition of $\Omega(\underline{X},\underline{A})^K$ in Proposition 2.5 to the factors that appear in the corresponding decomposition for $\Omega(\underline{X},\underline{A})^{K_I}$.

Lemma 2.6. *If $j\notin I$ then there is a commutative diagram*

$$\begin{tikzcd} (\underline{X},\underline{A})^K \arrow[r, "e_j"] \arrow[d, "\varphi_I"] & (\underline{X},\underline{A})^K \arrow[d, "\varphi_I"] \\ (\underline{X},\underline{A})^{K_I} \arrow[r, equal] & (\underline{X},\underline{A})^{K_I}. \end{tikzcd}$$

Proof. Recall that $I_j = [m]\backslash\{j\}$. Since $j\notin I$, K_I is a full subcomplex of K_{I_j}. So the projection $X^m\longrightarrow X^I$ is the same as the composite $X^m\longrightarrow X^{I_j}\longrightarrow X^m\longrightarrow X^I$. As in Lemma 2.3, the induced maps of polyhedral products $(\underline{X},\underline{A})^K\longrightarrow(\underline{X},\underline{A})^{K_I}$ and $(\underline{X},\underline{A})^K\longrightarrow(\underline{X},\underline{A})^{K_{I_j}}\longrightarrow(\underline{X},\underline{A})^K\longrightarrow(\underline{X},\underline{A})^{K_I}$ are the same. The lemma now follows. $\square$

Let $T(e_j)$ be the telescope of e_j. Taking telescopes of the horizontal maps in Lemma 2.6 immediately implies the following.

Corollary 2.7. *If $j\notin I$ then there is a commutative diagram*

$$\begin{tikzcd} (\underline{X},\underline{A})^K \arrow[r] \arrow[d, "\varphi_I"] & T(e_j) \arrow[d] \\ (\underline{X},\underline{A})^{K_I} \arrow[r, equal] & (\underline{X},\underline{A})^{K_I} \end{tikzcd}$$

$\square$

Now consider the map $f_{(a_1,\ldots,a_m)} = f_{a_1} \circ \cdots \circ f_{a_m}$ where f_{a_j} is either Ωe_j or $1 - \Omega e_j$. Suppose that $j \notin I$. If $a_j = 0$ then $f_{a_j} = \Omega e_j$. By Lemma 2.6, $\Omega\varphi_I \circ \Omega e_j = \Omega\varphi_I$. So as the idempotents $\{\Omega e_j \mid 1 \leq j \leq m\}$ commute, we obtain

$$\Omega\varphi_I \circ f_{a_1} \circ \cdots f_{a_{j-1}} \circ \Omega e_j \circ f_{a_{j+1}} \circ \cdots \circ f_{a_m} = \Omega\varphi_I \circ f_{a_1} \circ \cdots f_{a_{j-1}} \circ f_{a_{j+1}} \circ \cdots \circ f_{a_m}.$$

If $a_j = 1$ then $f_{a_j} = 1 - \Omega e_j$. By Corollary 2.7, $\Omega\varphi_I \circ (1 - \Omega e_j) \simeq *$. So as the idempotents $\{\Omega e_j \mid 1 \leq j \leq m\}$ commute, we obtain

$$\Omega\varphi_I \circ f_{a_1} \circ \cdots f_{a_{j-1}} \circ (1 - \Omega e_j) \circ f_{a_{j+1}} \circ \cdots \circ f_{a_m} \simeq *.$$

Doing this for every $j \notin I$ gives the following.

Lemma 2.8. *Let $I = \{i_1, \ldots, i_k\} \subseteq [m]$. The following hold:*

(a) *if $a_j = 0$ for every $j \notin I$ then $\Omega\varphi_I \circ f_{a_1} \circ \cdots \circ f_{a_m} \simeq \Omega\varphi_I \circ f_{a_{i_1}} \circ \cdots \circ f_{a_{i_k}}$;*

(b) *if $a_j = 1$ for some $j \notin I$ then $\Omega\varphi_I \circ f_{a_1} \circ \cdots \circ f_{a_m} \simeq *$.*

□

Let $\mathcal{J}_I$ be the index set for the $2^{|I|}$ idempotents $f_{(a_{i_1},\ldots,a_{i_k})}$ used to decompose $\Omega(\underline{X},\underline{A})^{K_I}$ in Theorem 2.5. Notice that by Lemma 2.8 (a), $f_{(a_{i_1},\ldots,a_{i_k})}$ corresponds precisely to the idempotent $f_{(a_1,\ldots,a_m)}$ on $\Omega(\underline{X},\underline{A})^K$ where every $j \notin I$ has $a_j = 0$. Let $T(a_{i_1},\ldots,a_{i_k})$ be the telescope of $f_{(a_{i_1},\ldots,a_{i_k})}$. Then Lemma 2.8 implies the following.

Lemma 2.9. *Let $I = \{i_1, \ldots, i_k\} \subseteq [m]$. The following hold:*

(a) *if $a_j = 0$ for every $j \notin I$ then there is a commutative diagram*

$$\begin{tikzcd}
\Omega(\underline{X},\underline{A})^K \arrow[r] \arrow[d, "\Omega\varphi_I"] & T(a_1,\ldots,a_m) \arrow[d, "\simeq"] \\
\Omega(\underline{X},\underline{A})^{K_I} \arrow[r] & T(a_{i_1},\ldots,a_{i_k});
\end{tikzcd}$$

(b) *if $a_j = 1$ for some $j \notin I$ then the composite $T(a_1,\ldots,a_m) \hookrightarrow \Omega(\underline{X},\underline{A})^K \xrightarrow{\Omega\varphi_I} \Omega(\underline{X},\underline{A})^{K_I}$ is null homotopic.*

□

From Lemma 2.9 we obtain compatibility for the decompositions of $\Omega(\underline{X},\underline{A})^K$ and $\Omega(\underline{X},\underline{A})^{K_I}$.

Proposition 2.10. *Let $I \subseteq [m]$. There is a homotopy commutative diagram*

$$\begin{tikzcd}
\Omega(\underline{X},\underline{A})^K \arrow[r, "\simeq"] \arrow[d, "\Omega\varphi_I"] & \prod_{(a_1,\ldots,a_m)\in\mathcal{J}} T(a_1,\ldots,a_m) \arrow[d, "\pi_I"] \\
\Omega(\underline{X},\underline{A})^{K_I} \arrow[r, "\simeq"] & \prod_{(a_{i_1},\ldots,a_{i_k})\in\mathcal{J}_I} T(a_{i_1},\ldots,a_{i_k})
\end{tikzcd}$$

where π_I projects away from factors with $a_j = 1$ for some $j \notin I$ and identifies $T(a_1, \ldots, a_m)$ with $T(a_{i_1}, \ldots, a_{i_k})$ for factors with $a_j = 0$ for all $j \notin I$. $\square$

Next, we bring in the dual polyhedral product. By definition,

$$(\underline{X}, \underline{A})^K_D = \operatorname*{\underleftarrow{holim}}_{I \subsetneq [m]} (\underline{X}, \underline{A})^{K_I}$$

where the homotopy inverse limit is taken over the maps of polyhedral products $(\underline{X}, \underline{A})^{K_J} \longrightarrow (\underline{X}, \underline{A})^{K_I}$ induced by the projection $X^J \longrightarrow X^I$ when $I \subseteq J$. Looping, we obtain

$$\Omega(\underline{X}, \underline{A})^K_D = \operatorname*{\underleftarrow{holim}}_{I \subsetneq [m]} \Omega(\underline{X}, \underline{A})^{K_I}.$$

On the other hand, by Proposition 2.10 the decompositions of the spaces $\Omega(\underline{X}, \underline{A})^{K_I}$ in Theorem 2.5 are compatible with the maps $\Omega(\underline{X}, \underline{A})^{K_I} \longrightarrow \Omega(\underline{X}, \underline{A})^{K_J}$ and induce projections onto factors. Therefore $\operatorname*{\underleftarrow{holim}}_{I \subsetneq [m]} \Omega(\underline{X}, \underline{A})^{K_I}$ is precisely the product of all possible distinct factors that appear in the decompositions of $\Omega(\underline{X}, \underline{A})^{K_I}$ for any $I \subsetneq [m]$. Put another way, the only factors of $\Omega(\underline{X}, \underline{A})^K$ which are not also factors of $\Omega(\underline{X}, \underline{A})^K_D$ are those that project trivially under every map π_I for all $I \subsetneq [m]$. There is only one such factor, $T(1, \ldots, 1)$, so we obtain the following.

Proposition 2.11. *There is a homotopy commutative diagram*

$$\begin{tikzcd}
\Omega(\underline{X}, \underline{A})^K \arrow[r, "\simeq"] \arrow[d, "\Omega\varphi"] & \prod_{(a_1, \ldots, a_m) \in \mathcal{J}} T(a_1, \ldots, a_m) \arrow[d, "\pi"] \\
\Omega(\underline{X}, \underline{A})^K_D \arrow[r, "\simeq"] & \prod_{(a_1, \ldots, a_m) \in \mathcal{J} \setminus (1, \ldots, 1)} T(a_1, \ldots, a_m)
\end{tikzcd}$$

where π is the projection. $\square$

Since every factor of $\Omega(\underline{X}, \underline{A})^K_D$ is also a factor of $\Omega(\underline{X}, \underline{A})^K$ and the only additional factor of $\Omega(\underline{X}, \underline{A})^K$ is $T(1, \ldots, 1)$, Proposition 2.11 immediately implies the following.

Theorem 2.12. *Assume that $(\underline{X}, \underline{A})^K$ is simply-connected. The map $\Omega(\underline{X}, \underline{A})^K \xrightarrow{\Omega\varphi} \Omega(\underline{X}, \underline{A})^K_D$ has a right homotopy inverse and there is a homotopy equivalence*

$$\Omega(\underline{X}, \underline{A})^K \simeq \Omega(\underline{X}, \underline{A})^K_D \times T(1, \ldots, 1).$$

$\square$

3. Further properties of the decompositions

Next, we show that each of the factors $T(a_1, \ldots, a_m)$ in the decompositions for $\Omega(\underline{X}, \underline{A})^K$ and $\Omega(\underline{X}, \underline{A})^K_D$ is a loop space. Define the space $F_{[m]}$ be the homotopy fibration

$$F_{[m]} \longrightarrow (\underline{X}, \underline{A})^K \xrightarrow{\varphi} (\underline{X}, \underline{A})^K_D.$$

Lemma 3.1. *There is a homotopy equivalence $T(1, \ldots, 1) \simeq \Omega F_{[m]}$.*

Proof. By Theorem 2.12, $\Omega\varphi$ has a right homotopy inverse. Thus, from the homotopy fibration defining $F_{[m]}$, we obtain a homotopy equivalence $\Omega(\underline{X},\underline{A})^K \simeq \Omega(\underline{X},\underline{A})^K_D \times \Omega F_{[m]}$. Comparing this to the homotopy decomposition in Theorem 2.12 we obtain $T(1,\ldots,1) \simeq \Omega F_{[m]}$. □

Fix a sequence $(a_1,\ldots,a_m) \in \mathcal{J}$. Let $\{a_{i_1},\ldots,a_{i_k}\}$ consist of all the elements in the sequence which are 1 and let $\{a_{j_1},\ldots,a_{j_\ell}\}$ consist of all the elements in the sequence which are 0. Let $I = \{i_1,\ldots,i_k\}$ and $J = \{j_1,\ldots,j_\ell\}$. Note that $k+\ell = m$ and $I \cap J = \emptyset$.

By Lemma 2.9 (a) there is a homotopy equivalence $T(a_1,\ldots,a_m) \simeq T(a_{i_1},\ldots,a_{i_k})$, where $T(a_{i_1},\ldots,a_{i_k})$ is the telescope of the idempotent $f_{a_{i_1}} \circ \cdots \circ f_{a_{i_k}} = (1-\Omega e_{i_1}) \circ \cdots \circ (1-\Omega e_{i_k})$ on $\Omega(\underline{X},\underline{A})^{K_I}$. Note that each a_{i_t} equals 1 for $1 \leq t \leq k$, so applying Lemma 3.1 to the case of $\Omega(\underline{X},\underline{A})^{K_I}$ immediately implies the following.

Lemma 3.2. *There is a homotopy equivalence*

$$T(a_1,\ldots,a_m) \simeq \Omega F_I$$

where $I = \{i_1,\ldots,i_k\}$ consists of those indices in $[m]$ for which $a_{i_t} = 1$ and F_I is the homotopy fibre of the map $(\underline{X},\underline{A})^{K_I} \longrightarrow (\underline{X},\underline{A})^{K_I}_D$. □

Consequently, the decompositions of $\Omega(\underline{X},\underline{A})^K$ in Theorem 2.5 and of $\Omega(\underline{X},\underline{A})^K_D$ in Theorem 2.11 can be rewritten as follows.

Theorem 3.3. *There is a homotopy commutative diagram*

$$\begin{tikzcd}
\Omega(\underline{X},\underline{A})^K \arrow[r, "\simeq"] \arrow[d, "\Omega\varphi"] & \prod_{I\subseteq[m]} \Omega F_I \arrow[d, "\pi"] \\
\Omega(\underline{X},\underline{A})^K_D \arrow[r, "\simeq"] & \prod_{I\subsetneq[m]} \Omega F_I.
\end{tikzcd}$$

where π is the projection. □

Remark 3.4. It is tempting to suspect that the homotopy decompositions in Theorem 3.3 deloop, that is, that there are homotopy equivalences $(\underline{X},\underline{A})^K \simeq \prod_{I\subseteq[m]} F_I$ and $(\underline{X},\underline{A})^K_D \simeq \prod_{I\subsetneq[m]} F_I$. But $(\underline{X},\underline{A})^K$ and $(\underline{X},\underline{A})^K_D$ are not H-spaces so any hope of delooping the homotopy equivalences would come from the homotopy fibrations $F_I \longrightarrow (\underline{X},\underline{A})^{K_I} \xrightarrow{\varphi} (\underline{X},\underline{A})^{K_I}_D$ having a splitting of the form $(\underline{X},\underline{A})^{K_I} \longrightarrow F_I$. But this does not happen even in the simplest cases. For example, let K be two disjoint points. By the definition of the polyhedral product, $(\underline{X},\underline{A})^K = (X_1 \times A_2) \cup (A_1 \times X_2)$. The proper full subcomplexes of $K = \{1\}\coprod\{2\}$ are $\{1\}$, $\{2\}$ and $\emptyset$. The corresponding polyhedral products are X_1, X_2 and $*$. Therefore, $(\underline{X},\underline{A})^K_D$ is the homotopy inverse limit of the diagram $X_1 \longrightarrow * \longleftarrow X_2$, which is $X_1 \times X_2$. This implies that space $F_{[2]}$ is the homotopy fibre of the inclusion $(X_1 \times A_1) \cup (A_1 \times X_2) \longrightarrow X_1 \times X_2$. Specializing to $A_1 = A_2 = *$, this fibre is homotopy equivalent to that of the inclusion $X_1 \vee X_2 \longrightarrow X_1 \times X_2$, which by the Hilton-Milnor Theorem, is

$\Omega X_1 * \Omega X_2$. The only case when the map $\Omega X_1 * \Omega X_2 \longrightarrow X_1 \vee X_2$ has a left homotopy inverse is when at least one of X_1 or X_2 is trivial.

4. Further refinement in the case of $(\underline{CX}, \underline{X})^K$

Recall from the Introduction that if the simplicial complex K is totally homology fillable (a property that includes shifted, shellable and sequentially Cohen-Macaulay complexes) then there is a homotopy equivalence

$$(\underline{CX}, \underline{X})^K \simeq \bigvee_{I \notin K} \Sigma |K_I| \wedge \widehat{X}^I$$

and $\Sigma|K_I|$ is homotopy equivalent to a wedge of spheres. Thus $(\underline{CX}, \underline{X})^K$ is homotopy equivalent to a wedge sum of spaces of the form $\Sigma^t X_{i_1} \wedge \cdots \wedge X_{i_k}$ for various $t \geq 1$ and $1 \leq i_1 < \cdots < i_k \leq m$. In this section we show that for this class of polyhedral products the spaces F_I that appear in the decompositions of $\Omega(\underline{X}, \underline{A})^K$ and $\Omega(\underline{X}, \underline{A})^K_D$ can be more explicitly identified.

To prepare, we require two general lemmas. For spaces B and C the *right half-smash* of B and C is the quotient space $B \rtimes C = (B \times C)/\sim$ where $(*, c) \sim *$. It is well-known that if B is a co-H-space then there is a homotopy equivalence $B \rtimes C \simeq B \vee (B \wedge C)$. The following lemma is well known and follows easily from the methods in [G2] (a more detailed statement and its proof appear later in Theorem 7.1).

Lemma 4.1. *Let B be a path-connected pointed space and C a simply-connected, pointed space. Let $B \vee C \longrightarrow C$ be the pinch map. Then there is a homotopy fibration*

$$B \rtimes \Omega C \xrightarrow{f} B \vee C \longrightarrow C.$$

This homotopy fibration is natural for maps $B \longrightarrow B'$ and $C \longrightarrow C'$ $\square$

In [J] it is shown that if Y is a pointed, path-connected space then there is a homotopy equivalence

$$\Sigma\Omega\Sigma Y \simeq \bigvee_{n=1}^{\infty} \Sigma Y^{(n)}.$$

An immediate consequence is the following.

Lemma 4.2. *Let X and Y be a pointed, path-connected spaces. Then there is a homotopy equivalence*

$$\Sigma X \wedge \Omega\Sigma Y \simeq \bigvee_{n=1}^{\infty} (\Sigma X \wedge Y^{(n)})$$

which is natural for maps $X \longrightarrow X'$ and $Y \longrightarrow Y'$. $\square$

We now give a construction that will identify the homotopy type of the space $F_{[m]}$ in the case of a polyhedral product $(\underline{CX}, \underline{X})^K$ wheere K is totally homology fillable. Recall that, for $1 \leq j \leq m$, $I_j = [m] \backslash \{j\}$ and e_j is the idempotent

$$e_j \colon (\underline{CX}, \underline{X})^K \longrightarrow (\underline{CX}, \underline{X})^{K_{I_j}} \longrightarrow (\underline{CX}, \underline{X})^K$$

induced by projecting $(CX)^m$ to $(CX)^{I_j}$ and then including back into $(CX)^m$. First consider e_1. Since $(\underline{CX}, \underline{X})^K$ is homotopy equivalent to a wedge of spaces of the form $\Sigma^t X_{i_1} \wedge \cdots \wedge X_{i_k}$ for various $t \geq 1$ and $1 \leq i_1 < \cdots < i_k \leq m$, we can write

$$(\underline{CX}, \underline{X})^K \simeq \Sigma B_1 \vee \Sigma C_1$$

where each wedge summand of B_1 has X_1 as a smash factor and each wedge summand of C_1 does not have X_1 as a smash factor.

Lemma 4.3. *The following hold:*

(a) *the restriction of e_1 to ΣB_1 is null homotopic;*

(b) *the restriction of e_1 to ΣC_1 is the inclusion of ΣC_1 into $(\underline{CX}, \underline{X})^K$;*

(c) *part (a) implies that there is a homotopy commutative diagram*

$$\begin{tikzcd} \Sigma B_1 \vee \Sigma C_1 \arrow[r, "q_1"] \arrow[d, "\simeq"] & \Sigma C_1 \arrow[d] \\ (\underline{CX}, \underline{X})^K \arrow[r] & (\underline{CX}, \underline{X})^{K_{I_1}} \end{tikzcd}$$

where q_1 is the pinch map.

Proof. The wedge decomposition of $(\underline{CX}, \underline{X})^K$ is natural with respect to maps of simplicial complexes. Applying this to the composite $e_1 \colon (\underline{CX}, \underline{X})^K \longrightarrow (\underline{CX}, \underline{X})^{K_{I_1}} \longrightarrow (\underline{CX}, \underline{X})^K$, the fact that $I_1 = \{2, \ldots, m\}$ implies that any wedge summand of $(\underline{CX}, \underline{X})^K$ involving X_1 is mapped trivially to $(\underline{CX}, \underline{X})^{K_1}$ while any wedge summand not involving X_1 is mapped identically to itself by e_1. This proves parts (a) and (b). Part (c) follows immediately from part (a). $\square$

Define the space G_1 and the map g_1 by the homotopy fibration

$$G_1 \xrightarrow{g_1} \Sigma B_1 \vee \Sigma C_1 \xrightarrow{q_1} \Sigma C_1.$$

By Lemmas 4.1 and 4.2 there are natural homotopy equivalences

$$G_1 \simeq \Sigma B_1 \rtimes \Omega\Sigma C_1 \simeq \Sigma B_1 \vee (\Sigma B_1 \wedge \Omega\Sigma C_1) \simeq \Sigma B_1 \vee \left(\bigvee_{n=1}^{\infty} (\Sigma B_1 \wedge (C_1)^{(n)})\right). \tag{4}$$

Observe that, by definition, each wedge summand in B_1 is a smash product with X_1 as a factor, so every wedge summand of $B_1 \wedge (C_1)^{(n)}$ is also a smash product with X_1 as a factor. Therefore, every wedge summand of G_1 is the suspension of a smash product that has X_1 as a factor. We now separate out those wedge summands that also have X_2 as a factor. As the wedge summands are all suspensions, we can write

$$G_1 = \Sigma B_2 \vee \Sigma C_2$$

where each wedge summand of B_2 is a smash product with X_1 and X_2 as factors and each wedge summand of C_2 is a smash product with X_1 as a factor but not X_2. Let ψ_1 be the composite

$$\psi_1\colon G_1 \xrightarrow{g_1} \Sigma B_1 \vee \Sigma C_1 \xrightarrow{\simeq} (\underline{CX}, \underline{X})^K.$$

Lemma 4.4. *For the composite* $G_1 = \Sigma B_2 \vee \Sigma C_2 \xrightarrow{\psi_1} (\underline{CX}, \underline{X})^K \xrightarrow{e_2} (\underline{CX}, \underline{X})^K$ *the following hold:*

(a) *the restriction of* $e_2 \circ \psi_1$ *to* ΣB_2 *is null homotopic;*

(b) *the restriction of* $e_2 \circ \psi_1$ *to* ΣC_2 *is homotopic to the restriction of* ψ_1 *to* ΣC_1*;*

(c) *part (a) implies that there is a homotopy commutative diagram*

$$\begin{tikzcd}
\Sigma B_2 \vee \Sigma C_2 \arrow[r, "q_2"] \arrow[d, "\simeq"] & \Sigma C_2 \arrow[dd] \\
G_1 \arrow[d, "\psi_1"] & \\
(\underline{CX}, \underline{X})^K \arrow[r] & (\underline{CX}, \underline{X})^{K_{I_2}}
\end{tikzcd}$$

where q_2 *is the pinch map.*

Proof. The proofs of parts (a) and (b) start by reorganizing the data in order to apply the naturality of Lemma 4.1. First consider $(\underline{CX}, \underline{X})^K \simeq \Sigma B_1 \vee \Sigma C_1$. As all the wedge summands of ΣB_1 and ΣC_1 are suspensions, we may write $\Sigma B_1 = \Sigma B_{1,1} \vee \Sigma B_{1,2}$ and $\Sigma C_1 = \Sigma C_{1,1} \vee \Sigma C_{1,2}$ where $\Sigma B_{1,1}$ ($\Sigma C_{1,1}$ respectively) consists of those wedge summands of ΣB_1 (ΣC_1) having X_1 as a smash factor but not X_2, and $\Sigma B_{1,2}$ ($\Sigma C_{1,2}$) consists of those wedge summands of ΣB_1 (ΣC_1) having both X_1 and X_2 as smash factors. The pinch map $\Sigma B_1 \vee \Sigma C_1 \xrightarrow{q_1} \Sigma C_1$ can then be rewritten as a pinch map $\Sigma B_{1,1} \vee \Sigma B_{1,2} \vee \Sigma C_{1,1} \vee \Sigma C_{1,2} \longrightarrow \Sigma C_{1,1} \vee \Sigma C_{1,2}$.

Let $\Sigma B_2' = \Sigma B_{1,2} \vee \Sigma C_{1,2}$ and let $\Sigma C_2' = \Sigma B_{1,1} \vee \Sigma C_{1,1}$. Notice that $(\underline{CX}, \underline{X})^K \simeq \Sigma B_2' \vee \Sigma C_2'$ where $\Sigma B_2'$ consists of all wedge summands in $(\underline{CX}, \underline{X})^K$ which are smash products with X_2 as a factor and $\Sigma C_2'$ consists of all wedge summands in $(\underline{CX}, \underline{X})^K$ which are smash products not having X_2 as a factor. As in Lemma 4.3, the restriction of e_2 to $\Sigma B_2'$ is null homotopic and the restriction to $\Sigma C_2'$ is the inclusion of $\Sigma C_2'$ into $(\underline{CX}, \underline{X})^K$. Therefore the composite $\Sigma B_2' \vee \Sigma C_2' \xrightarrow{\simeq} (\underline{CX}, \underline{X})^K \longrightarrow (\underline{CX}, \underline{X})^{K_{I_2}}$ factors through the pinch map $q_2'\colon \Sigma B_2' \vee \Sigma C_2' \longrightarrow \Sigma C_2'$. Reordering the wedge summands, q_2' can be regarded as the wedge sum $\Sigma B_{1,1} \vee \Sigma B_{1,2} \vee \Sigma C_{1,1} \vee \Sigma C_{1,2} \xrightarrow{q_B \vee q_C} \Sigma B_{1,1} \vee \Sigma C_{1,1}$ of the pinch maps $q_B\colon \Sigma B_1 = \Sigma B_{1,1} \vee \Sigma B_{1,2} \longrightarrow \Sigma B_{1,1}$ and $q_C\colon \Sigma C_1 = \Sigma C_{1,1} \vee \Sigma C_{1,2} \longrightarrow \Sigma C_{1,1}$.

Putting q_1 and q_2' together, by the naturality of Lemma 4.1 there is a homotopy fibration diagram

$$\begin{tikzcd}
\Sigma B_1 \rtimes \Omega\Sigma C_1 \arrow[r] \arrow[d, "q_B \rtimes \Omega q_C"] & \Sigma B_{1,1} \vee \Sigma B_{1,2} \vee \Sigma C_{1,1} \vee \Sigma C_{1,2} \arrow[r, "q_1"] \arrow[d, "q_B \vee q_C"] & \Sigma C_{1,1} \vee \Sigma C_{1,2} \arrow[d, "q_C"] \\
\Sigma B_{1,1} \rtimes \Omega\Sigma C_{1,1} \arrow[r] & \Sigma B_{1,1} \vee \Sigma C_{1,1} \arrow[r, "q"] & \Sigma C_{1,1}
\end{tikzcd}$$

where q is the pinch map. Putting the left square together with the factorization of e_2 through q_2' gives a homotopy commutative diagram

$$\begin{tikzcd}
\Sigma B_1 \rtimes \Omega\Sigma C_1 \arrow[r] \arrow[d, "q_B \rtimes \Omega q_C"] & \Sigma B_{1,1} \vee \Sigma B_{1,2} \vee \Sigma C_{1,1} \vee \Sigma C_{1,2} \arrow[r, equal] \arrow[d, "q_B \vee q_C"] & \Sigma B_2' \vee \Sigma C_2' \arrow[r, "\simeq"] \arrow[d, "q_2'"] & (\underline{CX}, \underline{X})^K \arrow[d] \\
\Sigma B_{1,1} \rtimes \Omega\Sigma C_{1,1} \arrow[r] & \Sigma B_{1,1} \vee \Sigma C_{1,1} \arrow[r, equal] & \Sigma C_2' \arrow[r] & (\underline{CX}, \underline{X})^{K_{I_2}}.
\end{tikzcd}$$

Notice that $G_1 \simeq \Sigma B_1 \rtimes \Omega\Sigma C_1$ and the top row is homotopic to ψ_1.

The maps q_B and q_C pinch out any wedge summands of ΣB_1 and ΣC_1 respectively that have X_2 as a smash factor. As q_B and q_C are suspensions, the naturality of the wedge decomposition (4) of $G_1 \simeq \Sigma B_1 \rtimes \Omega\Sigma C_1$ implies that $q_B \rtimes \Omega q_C$ pinches out any wedge summand of G_1 having both X_1 and X_2 as factors, and sends any wedge summand having X_1 as a smash factor but not X_2 identically to itself in $\Sigma B_{1,1} \rtimes \Omega\Sigma C_{1,1}$. That is, $q_B \rtimes \Omega q_C$ is the same as the map $G_1 = \Sigma B_2 \vee \Sigma C_2 \xrightarrow{q_2} \Sigma C_2$. Therefore, the homotopy commutativity of the previous diagram implies that the restriction of $e_2 \circ \psi_1$ to ΣB_2 is null homotopic and the restriction to ΣC_2 is the restriction of ψ_1. This proves parts (a) and (b). Part (c) follows immediately from part (a). □

Now proceed as before by taking the homotopy fibre of the pinch map $G_1 = \Sigma B_2 \vee \Sigma C_2 \xrightarrow{q_2} \Sigma C_2$. Iterating, for $1 \leq j < m$ we obtain homotopy fibrations

$$G_{j+1} \xrightarrow{g_{j+1}} G_j \simeq \Sigma B_j \vee \Sigma C_j \xrightarrow{q_j} \Sigma C_j$$

where q_j is the pinch map; every wedge summand of B_j is a smash product with $X_1, \ldots, X_j$ as factors; every wedge summand of C_j is a smash product with $X_1, \ldots, X_{j-1}$ as factors but not X_j; and there are natural homotopy equivalences

$$G_{j+1} \simeq \Sigma B_j \rtimes \Omega\Sigma C_j \simeq \Sigma B_j \vee (\Sigma B_j \wedge \Omega\Sigma C_j) \simeq \Sigma B_j \vee \left(\bigvee_{n=1}^{\infty} (\Sigma B_j \wedge (C_j)^{(n)})\right).$$

Further, if ψ_j is the composite

$$\psi_j \colon G_j \xrightarrow{g_j} G_{j-1} \xrightarrow{\psi_{j-1}} (\underline{CX}, \underline{X})^K$$

then arguing as in Lemma 4.4 the following hold.

Lemma 4.5. *For the composite $G_j = \Sigma B_j \vee \Sigma C_j \xrightarrow{\psi_j} (\underline{CX}, \underline{X})^K \xrightarrow{e_j} (\underline{CX}, \underline{X})^K$ the following hold:*

(a) *the restriction of $e_j \circ \psi_j$ to ΣB_j is null homotopic;*

(b) *the restriction of $e_j \circ \psi_j$ to ΣC_j is homotopic to the restriction of ψ_j to ΣC_j;*

(c) *part (a) implies that there is a homotopy commutative diagram*

$$\begin{tikzcd}
\Sigma B_j \vee \Sigma C_j \arrow[r, "q_j"] \arrow[d, "\simeq"] & \Sigma C_j \arrow[dd] \\
G_j \arrow[d, "\psi_j"] & \\
(\underline{CX}, \underline{X})^K \arrow[r] & (\underline{CX}, \underline{X})^{K_{I_j}}
\end{tikzcd}$$

where q_j is the pinch map.

□

For $1 \leq j \leq m$, let θ_j be the composite

$$\theta_j \colon \Sigma C_j \hookrightarrow \Sigma B_j \vee \Sigma C_j = G_j \xrightarrow{\psi_j} (\underline{CX}, \underline{X})^K$$

and let θ be the product of the maps $\Omega\theta_j$:

$$\theta \colon \prod_{j=1}^{m} \Omega\Sigma C_j \longrightarrow \Omega(\underline{CX}, \underline{X})^K.$$

Let $\theta \cdot \psi_m$ be the composite

$$\theta \cdot \psi_m \colon (\prod_{j=1}^{m} \Omega\Sigma C_j) \times \Omega G_m \xrightarrow{\theta \times \psi_m} \Omega(\underline{CX}, \underline{X})^K \times \Omega(\underline{CX}, \underline{X})^K \xrightarrow{\mu} \Omega(\underline{CX}, \underline{X})^K$$

where μ is the loop multiplication.

Lemma 4.6. *Suppose that K is totally homology fillable. Then the map $(\prod_{j=1}^{m} \Omega\Sigma C_j) \times \Omega G_m \xrightarrow{\theta\cdot\psi_m} \Omega(\underline{CX}, \underline{X})^K$ is a homotopy equivalence.*

Proof. In general, the homotopy fibration $B \rtimes \Omega C \longrightarrow B \vee C \longrightarrow C$ in Lemma 4.1 has a section $C \longrightarrow B \vee C$ given by the inclusion of the wedge summand. Therefore, after looping, there is a homotopy equivalence $\Omega C \times (\Omega B \rtimes \Omega C) \xrightarrow{\simeq} \Omega(B \vee C)$. In our case, for $1 \leq j < m$, from the homotopy fibration $G_{j+1} \xrightarrow{g_{j+1}} G_j = \Sigma B_j \vee \Sigma C_j \xrightarrow{q_j} \Sigma C_j$ we obtain a homotopy equivalence $\Omega\Sigma C_j \times \Omega G_{j+1} \xrightarrow{\simeq} \Omega G_j$. Iteratively substituting the homotopy equivalence for ΩG_{j+1} into that for ΩG_j, we obtain a homotopy equivalence $(\prod_{j=1}^{m} \Omega\Sigma C_j) \times \Omega G_m \xrightarrow{\simeq} \Omega(\underline{CX}, \underline{X})^K$. Notice that the restriction of this homotopy equivalence to $\Omega\Sigma C_j$ is the definition of $\Omega\theta_j$ and the restriction to ΩG_m is $\Omega\psi_m$. Thus the equivalence is $\theta \cdot \Omega\psi_m$. □

Theorem 3.3 and Lemma 4.6 give different homotopy equivalences for $\Omega(\underline{CX}, \underline{X})^K$. We wish to compare them.

Lemma 4.7. *The composite $G_m \xrightarrow{\psi_m} (\underline{CX}, \underline{X})^K \xrightarrow{\varphi} (\underline{CX}, \underline{X})^K_D$ is null homotopic.*

Proof. Fix an integer j for $1 \leq j < m$. Consider the homotopy fibration $G_{j+1} \xrightarrow{g_{j+1}} G_j \xrightarrow{q_j} \Sigma C_j$. By Lemma 4.5 (c), the composite $\gamma_j \colon G_j \xrightarrow{\psi_j} (\underline{CX}, \underline{X})^K \longrightarrow (\underline{CX}, \underline{X})^{K_{I_j}}$ factors through q_j. Therefore $\gamma_j \circ g_{j+1}$ is null homotopic. By definition, $\psi_{j+1} = \psi_j \circ g_{j+1}$, so the composite $G_{j+1} \xrightarrow{\psi_{j+1}} (\underline{CX}, \underline{X})^K \longrightarrow (\underline{CX}, \underline{X})^{K_{I_j}}$ is null homotopic. The recursive definition of ψ_m implies that it factors through ψ_j so the composite $G_m \xrightarrow{\psi_m} (\underline{CX}, \underline{X})^K \longrightarrow (\underline{CX}, \underline{X})^{K_{I_j}}$ is null homotopic. This holds for all $1 \leq j \leq m$, so ψ_m composes trivially into $(\underline{CX}, \underline{X})^{K_{I_j}}$ for all j. But this implies that ψ_m composes trivially to $(\underline{CX}, \underline{X})^{K_I}$ for every proper subset I of $[m]$ because the projection $(CX)^m \longrightarrow (CX)^I$ has to factor through some $(CX)^{I_t}$. Looping, $\Omega\psi_m$ composes trivially to $\Omega(\underline{CX}, \underline{X})^{K_I}$ for any $I \subsetneq [m]$. But by Proposition 2.11, $\Omega(\underline{CX}, \underline{X})^K_D$ decomposes as a product, each factor of which is a factor of $\Omega(\underline{CX}, \underline{X})^{K_I}$ for some I, and this decomposition is compatible with $\Omega\varphi$. Thus the composite $\Omega G_m \xrightarrow{\Omega\psi_m} \Omega(\underline{CX}, \underline{X})^K \xrightarrow{\Omega\varphi} \Omega(\underline{CX}, \underline{X})^K_D$ is null homotopic.

To deloop this, observe that as G_m is a suspension there is a map $G_m \longrightarrow \Sigma\Omega G_m$ which is a right homotopy inverse of the evaluation map $ev \colon \Sigma\Omega G_m \longrightarrow G_m$. The naturality of the evaluation map implies that there is a homotopy commutative diagram

$$\begin{tikzcd}
G_m \arrow[r] \arrow[dr, equal] & \Sigma\Omega G_m \arrow[r, "\Sigma\Omega\psi_m"] \arrow[d, "ev"] & \Sigma\Omega(\underline{CX}, \underline{X})^K \arrow[r, "\Sigma\Omega\varphi"] \arrow[d, "ev"] & \Sigma\Omega(\underline{CX}, \underline{X})^K_D \arrow[d, "ev"] \\
 & G_m \arrow[r, "\psi_m"] & (\underline{CX}, \underline{X})^K \arrow[r, "\varphi"] & (\underline{CX}, \underline{X})^K_D .
\end{tikzcd}$$

The null homotopy for $\Omega\varphi \circ \Omega\psi_m$ therefore implies that $\varphi \circ \psi_m$ is also null homotopic. $\square$

Recall that there is a homotopy fibration $F_{[m]} \longrightarrow (\underline{CX}, \underline{X})^K \xrightarrow{\varphi} (\underline{CX}, \underline{X})^K_D$.

Corollary 4.8. *There is a lift*

$$\begin{tikzcd}
 & G_m \arrow[dl, dashed, "\lambda"'] \arrow[d, "\psi_m"] & \\
F_{[m]} \arrow[r] & (\underline{CX}, \underline{X})^K \arrow[r, "\varphi"] & (\underline{CX}, \underline{X})^K_D
\end{tikzcd}$$

for some map λ. Further, $\Omega\lambda$ has a left homotopy inverse.

Proof. The existence of the lift follows immediately from Lemma 4.7. By Lemma 4.6, $\Omega\psi_m$ has a left homotopy inverse. The fact that λ factors through ψ_m then implies that $\Omega\lambda$ also has a left homotopy inverse. $\square$

Lemma 4.9. $\prod_{j=1}^m \Omega\Sigma C_j \xrightarrow{\theta} \Omega(\underline{CX}, \underline{X})^K \xrightarrow{\Omega\varphi} \Omega(\underline{CX}, \underline{X})^K_D$ *has a left homotopy inverse.*

Proof. Fix an integer j for $1 \leq j < m$. By Lemma 4.6, the map $\Omega\Sigma C_j \xrightarrow{\Omega\theta_j} \Omega(\underline{CX}, \underline{X})^K$ has a left homotopy inverse. On the other hand, by Lemma 4.5 (b), $e_j \circ \theta_j \simeq \theta_j$. This implies that the composite $\Omega\Sigma C_j \xrightarrow{\Omega\theta_j} \Omega(\underline{CX}, \underline{X})^K \longrightarrow \Omega(\underline{CX}, \underline{X})^{K_j} \longrightarrow \Omega(\underline{CX}, \underline{X})^K$ has a left homotopy inverse. Therefore the composite $\Omega\Sigma C_j \xrightarrow{\Omega\theta_j} \Omega(\underline{CX}, \underline{X})^K \longrightarrow \Omega(\underline{CX}, \underline{X})^{K_j}$ has a left homotopy inverse. By definition

of $(\underline{CX},\underline{X})^K_D$ as a homotopy limit, the map $(\underline{CX},\underline{X})^K \longrightarrow (\underline{CX},\underline{X})^{K_j}$ factors through $(\underline{CX},\underline{X})^K_D$. Thus the composite $\Omega\Sigma C_j \xrightarrow{\Omega\theta_j} \Omega(\underline{CX},\underline{X})^K \xrightarrow{\Omega\varphi} \Omega(\underline{CX},\underline{X})^K_D$ has a left homotopy inverse. This is true for all $1 \leq j \leq m$. Collectively, no overlap in the factors in the different retractions occurs because the product map $\prod_{j=1}^m \Omega\Sigma C_j \xrightarrow{\theta} \Omega(\underline{CX},\underline{X})^K$ has a left homotopy inverse. Hence the composite $\prod_{j=1}^m \Omega\Sigma C_j \xrightarrow{\theta} \Omega(\underline{CX},\underline{X})^K \xrightarrow{\Omega\varphi} \Omega(\underline{CX},\underline{X})^K_D$ also has a left homotopy inverse. □

Proposition 4.10. *Let K be a totally homology fillable simplicial complex. Then the map $G_m \xrightarrow{\lambda} F_{[m]}$ in Corollary 4.8 is a homotopy equivalence.*

Remark 4.11. It is worth repeating at this point that G_m is homotopy equivalent to a wedge of spaces of the form $\Sigma^t X_{i_1} \wedge \cdots \wedge X_{i_k}$ where $t \geq 1$ and $\{1,\ldots,m\} \subseteq \{i_1,\ldots,i_k\}$. That is, each X_i for $1 \leq i \leq m$ appears as a smash factor in every wedge summands of G_m.

Proof. On the one hand, by Theorem 3.3, the homotopy fibration $F_{[m]} \longrightarrow \Omega(\underline{CX},\underline{X})^K \xrightarrow{\Omega\varphi} \Omega(\underline{CX},\underline{X})^K_D$ splits to give a homotopy equivalence $\Omega(\underline{CX},\underline{X})^K \simeq \Omega(\underline{CX},\underline{X})^K_D \times \Omega F_{[m]}$. On the other hand, by Lemma 4.6, the composite $(\prod_{j=1}^m \Omega\Sigma C_j) \times \Omega G_m \xrightarrow{\theta\cdot\psi_m} \Omega(\underline{CX},\underline{X})^K$ is a homotopy equivalence. We compare the two decompositions.

By Lemma 4.9, the composite $h\colon \prod_{j=1}^m \Omega\Sigma C_j \xrightarrow{\theta} \Omega(\underline{CX},\underline{X})^K \xrightarrow{\Omega\varphi} \Omega(\underline{CX},\underline{X})^K_D$ has a left homotopy inverse. By Corollary 4.8, the map $\Omega G_m \xrightarrow{\Omega\psi_m} \Omega(\underline{CX},\underline{X})^K$ lifts to a map $\Omega G_m \xrightarrow{\Omega\lambda} \Omega F_{[m]}$ and $\Omega\lambda$ has a left homotopy inverse. Thus the product map

$$\Gamma\colon \Omega(\underline{CX},\underline{X})^K \xrightarrow{\ \simeq\ } (\prod_{j=1}^m \Omega\Sigma C_k) \times \Omega G_m \xrightarrow{h\times\Omega\lambda} \Omega(\underline{CX},\underline{X})^K_D \times \Omega F_{[m]} \xrightarrow{\ \simeq\ } \Omega(\underline{CX},\underline{X})^K$$

has the property that $h \times \Omega\lambda$ has a left homotopy inverse. In particular, $h \times \Omega\lambda$ induces an injection in homology and therefore Γ_* is an injection in homology with any coefficients. Taking $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Q}$ coefficients, Γ_* is a self-map of a finite type module which is an injection, and so it is an isomorphism. Thus Γ induces an isomorphism in mod-p and rational homology and so induces an isomorphism in integral homology. Therefore Γ is a homotopy equivalence. This implies that $h \times \Omega\lambda$ is a homotopy equivalence. As $h\times\Omega\lambda$ is a product map, each of h and $\Omega\lambda$ must therefore be a homotopy equivalence.

Finally, since $\Omega\lambda$ is a homotopy equivalence, it induces an isomorphism on homotopy groups, and therefore so does λ. Hence λ is also a homotopy equivalence, as asserted. □

By Theorem 3.3, each factor in the homotopy decompositions of $\Omega(\underline{CX},\underline{X})^K$ and $\Omega(\underline{CX},\underline{X})^K_D$ is of the form ΩF_I where F_I is the homotopy fibre of the map $(\underline{CX},\underline{X})^{K_I} \longrightarrow (\underline{CX},\underline{X})^{K_I}_D$. Further, by [IK2] any subcomplex of a totally homology fillable simplicial complex is itself totally homology fillable. Therefore Proposition 4.10 applies to each F_I to obtain the following.

Theorem 4.12. *Let K be a totally homology fillable simplicial complex. Then the homotopy decompositions*

$$\Omega(\underline{CX},\underline{X})^K \simeq \prod_{I\subset[m]} \Omega F_I \qquad \Omega(\underline{CX},\underline{X})^K_D \simeq \prod_{I\subsetneq[m]} \Omega F_I$$

in Theorem 3.3 have the property that each space F_I is homotopy equivalent to a wedge of summands, where each wedge summand is the suspension of a smash product having X_i as a factor for all $i \in I$. $\square$

5. A generalization to $(\underline{X}, \underline{*})^K$

In this section we show that Theorem 4.12 gives useful information for a wider range of polyhedral products. Let $\{X_i\}_{i=1}^m$ be a collection of pointed CW-complexes and let K be a simplicial complex on the vertex set $[m]$. Bahri, Bendersky, Cohen and Gitler [BBCG, Corollary 2.32], relying heavily on a result of Denham and Suciu [DS], show that there is a homotopy fibration

$$(\underline{C\Omega X}, \underline{\Omega X})^K \longrightarrow (\underline{X}, \underline{*})^K \longrightarrow \prod_{i=1}^m X_i. \tag{5}$$

The spaces $(\underline{X}, \underline{*})^K$ include many familiar spaces: if K is m disjoint points then $(\underline{X}, \underline{*})^K$ is homotopy equivalent to the wedge $X_1 \vee \cdots \vee X_m$; if K is the boundary of the standard $(m-1)$-simplex then $(\underline{X}, \underline{*})^K$ is the fat wedge in $\prod_{i=1}^m X_i$; and crucially to toric topology, if each $X_i = \mathbb{C}P^\infty$ then $(\underline{X}, \underline{*})^K$ is the Davis-Januszkiewicz space DJ_K.

Each vertex i of K is the full subcomplex $K_{\{i\}}$, and the polyhedral product $(\underline{X}, \underline{*})^{K_{\{i\}}}$ is simply X_i. So including i into K we obtain a map of polyhedral products $X_i \longrightarrow (\underline{X}, \underline{*})^K$ which, when composed to $\prod_{i=1}^m X_i$, is the inclusion of the i^{th} factor. After looping we can take the product of all such maps for $1 \leq i \leq m$ to obtain a section for the map $\Omega(\underline{X}, \underline{*})^K \longrightarrow \prod_{i=1}^m \Omega X_i$, implying the following.

Lemma 5.1. *The homotopy fibration (5) splits after looping, resulting in a homotopy equivalence*

$$\Omega(\underline{X}, \underline{*})^K \simeq (\prod_{i=1}^m \Omega X_i) \times \Omega(\underline{C\Omega X}, \underline{\Omega X})^K.$$

$\square$

We wish to show that the homotopy equivalence in Lemma 5.1 is compatible with that in Theorem 4.12. The following lemma does this. Write $F_I(\underline{X}, \underline{*})$ and $F_I(\underline{C\Omega X}, \underline{\Omega X})$ for the spaces F_I that appear in the respective decompositions of $\Omega(\underline{X}, \underline{*})^K$ and $\Omega(\underline{C\Omega X}, \underline{\Omega X})^K$ in Theorem 3.3.

Lemma 5.2. *Let $1 \leq i \leq m$. The following hold:*

(a) $F_{\{i\}}(\underline{X}, \underline{*}) \simeq X_i$;
(b) $F_{\{i\}}(\underline{C\Omega X}, \underline{\Omega X}) \simeq *$;
(c) *for $I \subseteq [m]$ and $I \neq \{i\}$ for any $1 \leq i \leq m$, the map $(\underline{C\Omega X}, \underline{\Omega X})^K \longrightarrow (\underline{X}, \underline{*})^K$ induces a homotopy equivalence $F_I(\underline{C\Omega X}, \underline{\Omega X}) \simeq F_I(\underline{X}, \underline{*})$.*

Proof. In general, for $(\underline{X}, \underline{A})^K$, the definition of the polyhedral product implies that $(\underline{X}, \underline{A})^{K_{\{i\}}} = X_i$. By the definition of the dual polyhedral product as a homotopy colimit over the proper subcomplexes of K, we obtain $(\underline{X}, \underline{A})_D^{K_{\{i\}}} = *$, and by definition, $F_{\{i\}}$ is the homotopy fibre of the map $(\underline{X}, \underline{A})^{K_{\{i\}}} \longrightarrow (\underline{X}, \underline{A})_D^{K_{\{i\}}}$. Thus $F_{\{i\}} \simeq X_i$. In particular, we obtain $F_{\{i\}}(\underline{X}, \underline{*}) \simeq X_i$ and $F_{\{i\}}(\underline{C\Omega X}, \underline{\Omega X}) \simeq C\Omega X_i \simeq *$. Part (c) now follows from parts (a) and (b) and Lemma 5.1. $\square$

Consequently, by applying Theorem 4.12 to $\Omega(\underline{C\Omega X},\underline{\Omega X})^K$ we obtain the following.

Theorem 5.3. *Let K be a totally homology fillable simplicial complex. Then there are homotopy decompositions*

$$\Omega(\underline{X},\underline{*})^K \simeq (\prod_{i=1}^m \Omega X_i) \times \prod_{\substack{I\subseteq[m] \\ I\neq\{i\}}} \Omega F_I \qquad \Omega(\underline{X},\underline{*})^K_D \simeq (\prod_{i=1}^m \Omega X_i) \times \prod_{\substack{I\subsetneq[m] \\ I\neq\{i\}}} \Omega F_I$$

where each F_I is homotopy equivalent to a wedge of summands, and each wedge summand is the suspension of a smash product having ΩX_i as a factor for all $i \in I$. □

In particular, if each X_i is $\mathbb{C}P^\infty$ then $(\underline{X},\underline{*})^K = DJ(K)$ and $\Omega X_i \simeq S^1$. So Theorem 5.3 gives a homotopy decomposition for $\Omega DJ(K)$ in which every F_I is homotopy equivalent to a wedge of spheres.

Part 2. Whitehead products.

6. Whitehead products and Porter's Theorem

In studying thin products it is important to have a good grip on the homotopy theory of the wedge $\bigvee_{i=1}^m X_i$. Consider the homotopy fibration

$$\Gamma(\underline{X}) \xrightarrow{\gamma(\underline{X})} \bigvee_{i=1}^m X_i \longrightarrow \prod_{i=1}^m X_i.$$

When $m = 2$ Ganea [G2] identified the homotopy type of $\Gamma(\underline{X})$ as $\Sigma\Omega X_1 \wedge \Omega X_2$ and the homotopy class of $\gamma(\underline{X})$ as a Whitehead product that depends on the evaluation maps from $\Sigma\Omega X_1$ and $\Sigma\Omega X_2$ to X_1 and X_2 respectively. Porter [P] generalized this by identifying the homotopy type of $\Gamma(\underline{X})$ for any $m \geq 2$.

Theorem 6.1. *Let $X_1, \ldots, X_m$ be simply-connnected, pointed CW-complexes. Then there is a homotopy equivalence*

$$\Gamma(\underline{X}) \simeq \bigvee_{k=2}^m \left(\bigvee_{1\leq i_1<\cdots<i_k\leq m} (k-1)(\Sigma\Omega X_{i_1} \wedge \cdots \wedge \Omega X_{i_k}) \right)$$

which is natural for maps $X_i \longrightarrow Y_i$. □

However, Porter did not identify the homotopy class of the map $\gamma(\underline{X})$. It is folklore that it too is determined by Whitehead products that depend on evaluation maps, but there seems to be no proof of this in the literature. As this will be important in what follows, in this section we carry out the identification. Along the way will be various constructions that will also play a role in the other sections in this part. As this section is long, we start with the statement we aim to prove.

For $1 \leq j \leq m$, let

$$s_j \colon X_j \longrightarrow \bigvee_{i=1}^m X_i$$

be the inclusion of the j^{th}-wedge summand. Let t_j be the composite

$$t_j\colon \Sigma\Omega X_j \xrightarrow{ev} X_j \xrightarrow{s_j} \bigvee_{i=1}^{m} X_i$$

where ev is the canonical evaluation map.

Theorem 6.2. *Let $X_1, \ldots, X_m$ be simply-connnected, pointed CW-complexes. Then there is a natural homotopy equivalence*

$$\Gamma(\underline{X}) \simeq \bigvee_{k=2}^{m}\left(\bigvee_{1\leq i_1<\cdots<i_k\leq m}(k-1)(\Sigma\Omega X_{i_1}\wedge\cdots\wedge\Omega X_{i_k})\right)$$

which may be chosen so that the restriction of $\gamma(\underline{X})$ to $\Sigma\Omega X_{i_1}\wedge\cdots\wedge\Omega X_{i_k}$ is an iterated Whitehead product of length k formed from the maps $t_{i_1},\ldots,t_{i_k}$.

Theorem 6.2 will be proved in two stages, first in the special case when each $X_i = \Sigma Y_i$ and then in the general case. The first case allows for the use of the Bott-Samelson Theorem. Take homology with field coefficients. There are algebra isomorphisms

$$H_*(\Omega(\bigvee_{i=1}^{m}\Sigma Y_i)) \cong T(V) \cong UL\langle V\rangle$$

where $V = \widetilde{H}_*(\bigvee_{i=1}^{m} Y_i)$, $T(\cdot)$ is the free tensor algebra functor, $L\langle V\rangle$ is the free Lie algebra on V and $UL\langle V\rangle$ is its universal enveloping algebra. The connection to Lie algebras plays a key role, so we begin with some properties of free Lie algebras.

6.1. **Lie algebra properties.** Let R be a principal ideal domain and let $V = \{v_1,\ldots,v_m\}$ be a free R-module. Let $L = L\langle V\rangle$ be the free Lie algebra generated by V. Let $L_{ab} = L_{ab}\langle V\rangle$ be the free abelian Lie algebra generated by V, that is, the Lie bracket in $L_{ab}\langle V\rangle$ is identically zero. There is a map of Lie algebras $L\longrightarrow L_{ab}$ defined by sending a generator $v_i\in L$ to $v_i\in L_{ab}$ and any nontrivial bracket in L to zero. Let $[L,L]$ be the kernel of this map. Then there is a short exact sequence of Lie algebras

$$0\longrightarrow [L,L]\longrightarrow L\longrightarrow L_{ab}\longrightarrow 0$$

and $[L,L]$ is a sub-Lie algebra of a free Lie algebra so is itself free. We will describe a Lie basis for $[L,L]$.

For any Lie algebra $\mathcal{L}$ with bracket $[\ ,\ ]$, if $x,y\in\mathcal{L}$ let $ad(x)(y)=[x,y]$. More generally, starting with $ad^0(x)(y)=y$, for any integer $k\geq 1$ let $ad^k(x)(y)=[x,ad^{k-1}(x)(y)]$.

Consider $L=L\langle v_1,\ldots,v_m\rangle$. For $1\leq i\leq m$, let $L_{ab}\langle v_i\rangle$ be the free abelian Lie algebra generated by $\{v_i\}$. An R-module basis for $L_{ab}\langle v_i\rangle$ is $\{1,v_i\}$. Define L_1 by the short exact sequence of Lie algebras

$$0\longrightarrow L_1\longrightarrow L\xrightarrow{\ell_1} L_{ab}\langle v_1\rangle\longrightarrow 0$$

where ℓ_1 sends 1 and v_1 to themselves and every other R-module basis element of L to zero. Then L_1 is the kernel of ℓ_1, it is a sub-Lie algebra of the free Lie algebra L, and so is itself free. One way to obtain a Lie basis for L_1 is as follows (see, for example, [CMN, 3.13] or [N, 8.7.2 and 8.7.3]). Let W_1 be a Lie basis for L_1. The Lie bracket $L\times L\longrightarrow L$ induces a right action $W_1\times\{v_1\}\longrightarrow W_1$ given by sending the pair (w,v_1) to $ad(w)(v_1)=[w,v_1]$. Let $\overline{W}_1=V-\{v_1\}=\{v_2,\ldots,v_m\}$. Restricting the action to the subset $\overline{W}_1$ we obtain

$$W_1=\{ad^k(x)(v_1)\mid x\in\overline{W}_1,k\geq 0\}.$$

Now iterate. For $t\leq i-1$, assume that L_t has been defined, it is a free Lie algebra with basis W_t, and $\{v_{t+1},\ldots,v_m\}\subseteq W_t$. Define L_i by the short exact sequence of Lie algebras

$$0\longrightarrow L_i\longrightarrow L_{i-1}\xrightarrow{\ell_i} L_{ab}\langle v_i\rangle\longrightarrow 0$$

where ℓ_i sends 1 and v_i to themselves and every other R-module basis element of L_{i-1} to zero. Then L_i is the kernel of ℓ_i, it is a sub-Lie algebra of the free Lie algebra L_{i-1}, and so is itself free. As above, if W_i is a Lie basis for L_i and $\overline{W}_{i-1}=W_{i-1}-\{v_i\}$, then

$$W_i=\{ad^k(x)(v_i)\mid x\in\overline{W}_{i-1},k\geq 0\}.$$

At the end of the iteration we obtain a free Lie algebra L_m with Lie basis W_m which depends recursively on W_{m-1}. The Lie algebra L_m is useful because, as the next lemma shows, it is identical to $[L,L]$, and so gives a way of describing the Lie basis of $[L,L]$.

Lemma 6.3. *There is an isomorphism of Lie algebras $L_m\cong[L,L]$.*

Proof. Observe that there is a commutative diagram of exact sequences

$$\begin{tikzcd}
L_1 \arrow[r] \arrow[d, "\ell_2"] & L \arrow[r, "\ell_1"] \arrow[d] & L_{ab}\langle v_1\rangle \arrow[d, equal] \\
L_{ab}\langle v_2\rangle \arrow[r] & L_{ab}\langle v_1\rangle\times L_{ab}\langle v_2\rangle \arrow[r] & L_{ab}\langle v_1\rangle .
\end{tikzcd}$$

This implies that the kernel of $L\longrightarrow L_{ab}\langle v_1\rangle\times L_{ab}\langle v_2\rangle$ is the same as that of ℓ_2, which is L_2. We now argue by induction. Assume that L_i is the kernel of the map $L\longrightarrow\prod_{t=1}^{i}L_{ab}\langle v_i\rangle$. Then there is a commutative diagram of exact sequences

$$\begin{tikzcd}
L_i \arrow[r] \arrow[d, "\ell_{i+1}"] & L \arrow[r] \arrow[d] & \prod_{t=1}^{i}L_{ab}\langle v_i\rangle \arrow[d, equal] \\
L_{ab}\langle v_{i+1}\rangle \arrow[r] & \prod_{t=1}^{i+1}L_{ab}\langle v_t\rangle \arrow[r] & \prod_{t=1}^{i}L_{ab}\langle v_t\rangle .
\end{tikzcd}$$

Therefore the kernel of $L \longrightarrow \prod_{t=1}^{i+1} L_{ab}\langle v_t\rangle$ is the same as that of ℓ_{i+1}, which is L_{i+1}. Hence, by induction, there is a short exact sequence of Lie algebras

$$L_m \longrightarrow L \longrightarrow \prod_{i=1}^{m} L_{ab}\langle v_i\rangle.$$

Observe that $\prod_{i=1}^{m} L_{ab}\langle v_i\rangle \cong L_{ab}\langle v_1,\ldots,v_m\rangle$ and $L \longrightarrow \prod_{i=1}^{m} L_{ab}\langle v_i\rangle$ is the abelianization. Therefore L_m is the kernel of abelianization map $L \longrightarrow L_{ab}$. But, by definition, this kernel is $[L,L]$. $\square$

We next describe a distinguished subset of the Lie basis for $[L,L]$.

Lemma 6.4. *In the Lie basis W_m for $[L,L]$, where $L = L\langle v_1,\ldots,v_m\rangle$, there are $m-1$ basis elements that are length m brackets having each v_i appear exactly once.*

Proof. The proof is by induction on m. The base case is when $m=2$, in which case $L = L\langle v_1,v_2\rangle$ and $[L,L]$ has only one bracket of length 2 having each v_i appear once, namely, $[v_1,v_2]$. Assume that the statement holds for $L\langle v_1,\ldots,v_{m-1}\rangle$.

Consider $L = L\langle v_1,\ldots,v_m\rangle$. By Lemma 6.3 there is a Lie algebra isomorphism $[L,L] \cong L_m$ where L_m is a free Lie algebra with Lie basis W_m. Recall that W_m has been defined recursively, $W_m = \overline{W}_{m-1} \cup \{ad^k(x)(v_m) \mid x \in \overline{W}_{m-1}, k \geq 1\}$, where $\overline{W}_{m-1} = W_{m-1} - \{v_m\}$. Observe that as we seek brackets where each v_i appears exactly once, we may restrict to the $k=1$ case. Define W'_m recursively by $W'_2 = \{v_i, ad(v_i)(v_1) \mid 2 \leq i \leq m\}$ and

$$W'_m = \overline{W}'_{m-1} \cup \{ad(x)(v_m) \mid x \in \overline{W}'_{m-1}\},$$

where $\overline{W}'_{m-1} = W'_{m-1} - \{v_m\}$. Since the maximum bracket length in W'_1 is 2, the recursive definition of W'_m implies that the maximum bracket length in W'_m is $m+1$.

Let w be a Lie basis element of length m in W'_m where each v_i appears once. First, suppose that w is in $\overline{W}'_{m-1}$. Since $W'_{m-1} = \overline{W}'_{m-2} \cup \{ad(x)(v_{m-1}) \mid x \in \overline{W}'_{m-2}\}$ and the maximum bracket length in $\overline{W}'_{m-2} \subset W'_{m-2}$ is $m-1$, we have $w \in \{ad(x)(v_{m-1}) \mid x \in \overline{W}'_{m-2}\}$. But then x must have bracket length $m-1$, and we may iterate the argument to show that $x = [\ldots,[y,v_1],v_2],\ldots,v_{m-1}]$ for $y \in W'_1$. Therefore $w = [\ldots,[y,v_1],v_2],\ldots,v_{m-1}],v_m]$ and the only choice for y which gives a bracket of length m with each v_i appearing once is $y = v_m$. Second, suppose that w is in $\{ad(x)(v_m) \mid x \in \overline{W}'_{m-1}\}$. Then x must be a Lie basis element where each v_i for $1 \leq i \leq m-1$ appears exactly once. As only the variables $v_1,\ldots,v_{m-1}$ appear, $x \in L\langle v_1,\ldots,v_m\rangle$, and by inductive hypothesis there are $m-2$ choices for such an x. Hence, in total there are $m-1$ Lie basis elements of $[L,L]$ of length m where each v_i appears once. $\square$

Let $\mathcal{B}$ consist of those Lie basis elements for $[L,L]$ which are length k brackets which consist of k different v_i's, for $2 \leq k \leq m$.

Lemma 6.5. *The set $\mathcal{B}$ decomposes as the disjoint union*

$$\coprod_{k=2}^{m}\left(\coprod_{1\leq i_1<\cdots i_k\leq m}\mathcal{B}_{i_1,\ldots,i_k}\right)$$

where $\mathcal{B}_{i_1,\ldots,i_k}$ consists of the $(k-1)$ distinct Lie basis elements in $L\langle v_{i_1},\ldots,v_{i_k}\rangle$ of length k where each v_{i_t} appears exactly once.

Proof. This follows by taking all possible ordered subsets $\{i_1,\ldots,i_k\}$ of $\{1,\ldots,m\}$ and applying Lemma 6.4 to $L\langle v_{i_1},\ldots,v_{i_k}\rangle$. $\square$

6.2. **The case when $X_i=\Sigma Y_i$ for $1\leq i\leq m$.** The set $\mathcal{B}$ will be used to define certain Whitehead products. Let $\underline{X}$ be a sequence of pointed, simply-connected topological spaces $X_1,\ldots,X_m$. As before, for $1\leq j\leq m$, let

$$s_j\colon X_j\longrightarrow\bigvee_{i=1}^{m}X_i$$

be the inclusion of the j^{th}-wedge summand and let t_j be the composite

$$t_j\colon\Sigma\Omega X_j\xrightarrow{ev}X_j\xrightarrow{s_j}\bigvee_{i=1}^{m}X_i$$

where ev is the canonical evaluation map. Let $w\in\mathcal{B}_{i_1,\ldots,i_k}\subset\mathcal{B}$. Let

$$\Theta_w(\underline{X})\colon\Sigma\Omega X_{i_1}\wedge\cdots\wedge\Omega X_{i_k}\longrightarrow\bigvee_{i=1}^{m}X_i$$

be the iterated Whitehead product of the maps $t_{i_1},\ldots,t_{i_k}$ corresponding to w. For example, if $w=[\ldots,[v_{i_1},v_{i_2}],v_{i_3}],\ldots,v_{i_k}]$ then $\Theta_w(\underline{X})=[\ldots,[t_{i_1},t_{i_2}],t_{i_3}],\ldots,t_{i_k}]$. Let

$$\Theta(\underline{X})\colon\bigvee_{k=2}^{m}\left(\bigvee_{1\leq i_1<\cdots<i_k\leq m}(k-1)\,\Sigma\Omega X_{i_1}\wedge\Omega X_{i_k}\right)\longrightarrow\bigvee_{i=1}^{m}X_i$$

be the wedge sum of the maps $\Theta_w(\underline{X})$ for all $w\in\mathcal{B}$.

Consider the homotopy fibration $\Gamma(\underline{X})\xrightarrow{\gamma(\underline{X})}\bigvee_{i=1}^{m}X_i\longrightarrow\prod_{i=1}^{m}X_i$. Every $\Theta_w(\underline{X})$ composes trivially into the product and therefore so does $\Theta(\underline{X})$. Thus there is a lift

$$\begin{tikzcd}
 & \bigvee_{k=2}^{m}\left(\bigvee_{1\leq i_1<\cdots<i_k\leq m}(k-1)\,\Sigma\Omega X_{i_1}\wedge\Omega X_{i_k}\right) \arrow[dl, dashed, "\theta"'] \arrow[d, "\Theta(\underline{X})"] \\
\Gamma(\underline{X}) \arrow[r, "\gamma(\underline{X})"] & \bigvee_{i=1}^{m}X_i
\end{tikzcd} \tag{6}$$

for some map θ. Notice that by Theorem 6.1, $\Gamma(\underline{X})$ has the same homotopy type as the domain of θ. We claim that θ is a homotopy equivalence. To show this we break the proof into two cases: the special case when each X_i is a suspension, and then the general case. We begin with some observations and definitions.

The adjoint of the map t_j is the composite $\Omega X_j \xrightarrow{E} \Omega\Sigma\Omega X_j \xrightarrow{\Omega ev} \Omega X \xrightarrow{\Omega s_j} \Omega(\bigvee_{i=1}^m X_i)$. Since $\Omega ev \circ E$ is homotopic to the identity map on ΩX, the adjoint of t_j is homotopic to Ωs_j. Let

$$S_w(\underline{X})\colon \Omega X_{i_1} \wedge \cdots \wedge \Omega X_{i_k} \longrightarrow \Omega(\bigvee_{i=1}^m X_i)$$

be the adjoint of $\Theta_w(\underline{X})$. Then $S_w(\underline{X})$ is an iterated Samelson product of the maps $\Omega s_{i_1}, \ldots, \Omega s_{i_k}$. Let

$$S(\underline{X})\colon \bigvee_{k=2}^m \left(\bigvee_{1\leq i_1<\cdots<i_k\leq m} (k-1)\,\Omega X_{i_1} \wedge \Omega X_{i_k} \right) \longrightarrow \Omega(\bigvee_{i=1}^m X_i)$$

be the wedge sum of the maps $S_w(\underline{X})$ for all $w \in \mathcal{B}$. Then $S(\underline{X})$ is the adjoint of $\Theta(\underline{X})$. By the James construction [J], the map $S(\underline{X})$ extends to a map

$$\Omega\Sigma \left(\bigvee_{k=2}^m \left(\bigvee_{1\leq i_1<\cdots<i_k\leq m} (k-1)\,\Omega X_{i_1} \wedge \Omega X_{i_k} \right)\right) \longrightarrow \Omega(\bigvee_{i=1}^m X_i) \tag{7}$$

which is homotopic to $\Omega\Theta(\underline{X})$.

Proposition 6.6. *Suppose that $X_i = \Sigma Y_i$ for $1 \leq i \leq m$. Then the map θ in (6) is a homotopy equivalence.*

Proof. It suffices to show that $\Omega\theta$ induces an isomorphism in homology with coefficients in any field. For if so, then $\Omega\theta$ induces an isomorphism in homology with integral coefficients and so is a homotopy equivalence by Whitehead's Theorem. As $\Omega\theta$ is a homotopy equivalence it induces an isomorphism on homotopy groups and therefore so does θ, implying that θ is a homotopy equivalence.

Take homology with field coefficients. By the Bott-Samelson Theorem there is an algebra isomorphism $H_*(\Omega(\bigvee_{i=1}^m \Sigma Y_i)) \cong T(\widetilde{H}_*(\bigvee_{i=1}^m Y_i))$. For $1 \leq i \leq m$, let $V_i = \widetilde{H}_*(Y_i)$ and let $V = \oplus_{i=1}^m V_i$. Then $H_*(\Omega(\bigvee_{i=1}^m \Sigma Y_i)) \cong T(V) \cong UL\langle V\rangle$. Let $w \in \mathcal{B}_{i_1,\ldots,i_k}$ and for $1 \leq t \leq k$ suppose that $x_{i_t} \in H_*(\Omega\sigma Y_{i_t}) \cong UL\langle V_{i_t}\rangle$. Write $w(x_{i_1}, \ldots, x_{i_k})$ for the bracket in $UL\langle V_{i_1} \oplus \cdots \oplus V_{i_k}\rangle$ in which each v_{i_t} in w is replaced by x_{i_t}. For example, if $w = [\ldots[v_{i_1}, v_{i_2}], v_{i_3}], \ldots v_{i_k}]$ then $w(x_{i_1}, \ldots, x_{i_k}) = [\ldots[x_{i_1}, x_{i_2}], x_{i_3}], \ldots, x_{i_k}]$. Since $\Omega\Sigma Y_j \xrightarrow{\Omega s_j} \Omega(\bigvee_{i=1}^m \Sigma Y_i)$ induces in homology the inclusion $UL\langle V_j\rangle \longrightarrow UL\langle V\rangle$ for all j, the Samelson product $\Omega\Sigma Y_{i_1} \wedge \cdots \wedge \Omega\Sigma Y_{i_k} \xrightarrow{S_w(\underline{\Sigma Y})} \Omega(\bigvee_{i=1}^m \Sigma Y_i)$ has image in homology given by

$$\begin{aligned} \mathrm{Im}(S_w(\underline{\Sigma Y}))_* &= \{w(x_{i_1}, \ldots, x_{i_k}) \mid x_{i_t} \in UL\langle V_{i_t}\rangle\} \\ &\subset UL\langle V_{i_1}\rangle \amalg \cdots \amalg UL\langle V_{i_k}\rangle \\ &= UL\langle V_{i_1} \oplus \cdots \oplus V_{i_k}\rangle \\ &= UL\langle V\rangle \end{aligned}$$

where $\amalg$ is the free product. In particular, as the x_{i_t}'s are in distinct free product factors, $(S_w(\underline{\Sigma Y}))_*$ is an injection. Further, any two distinct elements $w_1, w_2 \in \mathcal{B}_{i_1,\ldots,i_k}$ are distinct Lie basis elements in $L\langle v_{i_1}, \ldots, v_{i_k}\rangle$ so the images of $(S_{w_1}(\underline{\Sigma Y}))_*$ and $(S_{w_2}(\underline{\Sigma Y}))_*$ correspond to distinct bracketings

of elements in the free product $UL\langle V_{i_1}\rangle \amalg \cdots \amalg UL\langle V_{i_k}\rangle$, and therefore intersect trivially. More generally, the same holds for any two distinct elements in $\mathcal{B}$ so we obtain that the wedge sum $S(\underline{\Sigma Y})$ of the maps $S_w(\underline{\Sigma Y})$ induces an injection in homology. That is, if

$$M = \widetilde{H}_*(\bigvee_{k=1}^{m}\left(\bigvee_{1\leq i_1<\cdots<i_k\leq m}(k-1)\,\Omega\Sigma Y_{i_1}\wedge\Omega\Sigma Y_{i_k}\right))$$

then $(S(\underline{\Sigma Y}))_*$ maps M injectively into $UL\langle V\rangle \cong T(V)$. In general, since $T(V)$ is a free tensor algebra, any injection $M \longrightarrow T(V)$ from an R-module M extends to an injection of algebras $T(M) \longrightarrow T(V)$. Thus, in our case, extending the map $S(\underline{\Sigma Y})$ multiplicatively to $\Omega\Theta(\underline{\Sigma Y})$ by the James construction as in (7) and taking homology, we obtain an injection of algebras $(\Omega\Theta(\underline{\Sigma Y}))_*\colon T(M) \longrightarrow T(V)$. The homotopy commutativity of (6) therefore implies that $(\Omega\theta)_*$ is also an injection in homology, as required. $\square$

Proof of Theorem 6.2 when $X_i = \Sigma Y_i$ for $1 \leq i \leq m$. The homotopy equivalence for $\Gamma(\underline{\Sigma Y})$ and the description of $\gamma(\underline{\Sigma Y})$ in terms of Whitehead products follows immediately from the definition of $\Theta(\underline{\Sigma Y})$, the lift in (6) and Proposition 6.6. The naturality statement holds by the naturality of Whitehead products. $\square$

6.3. **The general case.** The idea is to consider the wedge of evaluation maps $\bigvee_{i=1}^{m}\Sigma\Omega X_i \longrightarrow \bigvee_{i=1}^{m} X_i$, apply the suspension case to the left wedge and use various naturality properties to deduce the general case for the right wedge. This will involve looping at a certain point, so it is useful to record the following lemma.

Recall that $s_j\colon X_j \longrightarrow \bigvee_{i=1}^{m} X_i$ is the inclusion of the j^{th} wedge summand. After looping the maps Ωs_j can be multiplied together to obtain a right homotopy inverse for the map $\Omega(\bigvee_{i=1}^{m} X_i) \longrightarrow \prod_{i=1}^{m}\Omega X_i$. Thus the homotopy fibration $\Gamma(\underline{X}) \xrightarrow{\gamma(\underline{X})} \bigvee_{i=1}^{m} X_i \longrightarrow \prod_{i=1}^{m} X_i$ splits after looping and we obtain the following.

Lemma 6.7. *The map $\Omega\Gamma(\underline{X}) \xrightarrow{\Omega\gamma(\underline{X})} \Omega(\bigvee_{i=1}^{m} X_i)$ has a left homotopy inverse.* $\square$

To compress notation, let $\mathcal{I}$ be an index set running over all sequences $(i_1,\ldots,i_k)$ where $2 \leq k \leq m$ and $1 \leq i_1 < \cdots < i_k \leq m$. Then

$$\bigvee_{k=2}^{m}\left(\bigvee_{1\leq i_1<\cdots<i_k\leq m}(k-1)(\Sigma\Omega X_{i_1}\wedge\cdots\wedge\Omega X_{i_k})\right) = \bigvee_{\alpha\in\mathcal{I}}(k-1)\,(\Sigma\Omega X_{i_1}\wedge\cdots\wedge\Omega X_{i_k}).$$

For $1 \leq i \leq m$, let

$$ev_i\colon \Sigma\Omega X_i \longrightarrow Xi$$

be the evaluation map and let

$$E_i\colon \Omega X_i \longrightarrow \Omega\Sigma\Omega X_i$$

be the suspension map.. Note that ev_i the left adjoint of the identity map on ΩX_i and E_i is the right adjoint of the identity map on $\Sigma\Omega X_i$. So $\Omega ev_i \circ E_i$ is homotopic to the identity map on ΩX_i. Let

$$\xi\colon \bigvee_{\alpha\in\mathcal{I}} (k-1)\,(\Sigma\Omega X_{i_1}\wedge\cdots\wedge\Omega X_{i_k}) \longrightarrow \bigvee_{\alpha\in\mathcal{I}} (k-1)\,(\Sigma\Omega\Sigma\Omega X_{i_1}\wedge\cdots\wedge\Omega\Sigma\Omega X_{i_k})$$

be the wedge sum of the maps $\Sigma E_{i_1}\wedge\cdots\wedge E_{i_k}$ and let

$$\zeta\colon \bigvee_{\alpha\in\mathcal{I}} (k-1)(\Sigma\Omega\Sigma\Omega X_{i_1}\wedge\cdots\wedge\Omega\Sigma\Omega X_{i_k}) \longrightarrow \bigvee_{\alpha\in\mathcal{I}} (k-1)\,(\Sigma\Omega X_{i_1}\wedge\cdots\wedge\Omega X_{i_k})$$

be the wedge sum of the maps $\Sigma\Omega ev_{i_1}\wedge\cdots\wedge\Omega ev_{i_k}$. The fact that $\Omega ev_i\circ E_i$ is homotopic to the identity map on ΩX_i immediately implies the following.

Lemma 6.8. *The map $\zeta\circ\xi$ is homotopic to the identity map on $\bigvee_{\mathcal{I}}(k-1)\,(\Sigma\Omega X_{i_1}\wedge\cdots\wedge\Omega X_{i_k})$.* $\square$

Proposition 6.9. *For any simply-connected spaces $X_1,\ldots,X_m$, the map θ in (6) is a homotopy equivalence.*

Proof. The naturality of the homotopy equivalence for $\Gamma(\underline{X})$ in Theorem 6.1 implies that there is a homotopy commutative diagram

$$\begin{tikzcd}
\bigvee_{\mathcal{I}}(k-1)(\Sigma\Omega\Sigma\Omega X_{i_1}\wedge\cdots\wedge\Omega\Sigma\Omega X_{i_k}) \arrow[r, "\gamma(\underline{\Sigma\Omega X})"] \arrow[d, "\zeta"] & \bigvee_{i=1}^m \Sigma\Omega X_i \arrow[d, "\bigvee_{i=1}^m ev_i"] \\
\bigvee_{\mathcal{I}}(k-1)(\Sigma\Omega X_{i_1}\wedge\cdots\wedge\Omega X_{i_k}) \arrow[r, "\gamma(\underline{X})"] & \bigvee_{i=1}^m X_i.
\end{tikzcd}$$

By the naturality of the Whitehead product there is also a homotopy commutative diagram

$$\begin{tikzcd}
\bigvee_{\mathcal{I}}(k-1)\Sigma\Omega\Sigma\Omega X_{i_1}\wedge\cdots\wedge\Omega\Sigma\Omega X_{i_k} \arrow[r, "\Theta(\underline{\Sigma\Omega X})"] \arrow[d, "\xi"] & \bigvee_{i=1}^m \Sigma\Omega X_i \arrow[d, "\bigvee_{i=1}^m ev_i"] \\
\bigvee_{\mathcal{I}}(k-1)\Sigma\Omega X_{i_1}\wedge\cdots\wedge\Omega X_{i_k} \arrow[r, "\Theta(\underline{X})"] & \bigvee_{i=1}^m X_i.
\end{tikzcd}$$

To further compress the notation, let

$$W(\underline{X}) = \bigvee_{\mathcal{I}}(k-1)(\Sigma\Omega X_{i_1}\wedge\cdots\wedge\Omega X_{i_k}).$$

Consider the diagram

$$
\begin{tikzcd}
& & & & \Omega W(\underline{X}) \arrow[d, "\Omega\xi"] \\
\Omega W(\underline{\Sigma\Omega X}) \arrow[rrrr, "\Omega\theta"] \arrow[dd, "\Omega\zeta"] \arrow[drr, "\Omega\Theta(\underline{\Sigma\Omega X})"] & & & & \Omega W(\underline{\Sigma\Omega X}) \arrow[dll, "\Omega\gamma(\underline{\Sigma\Omega X})"'] \arrow[dd, "\Omega\zeta"] \\
& & \Omega(\bigvee_{i=1}^{m}\Sigma\Omega X_i) \arrow[dd, "\Omega(\bigvee_{i=1}^{m} ev_i)"] & & \\
\Omega W(\underline{X}) \arrow[drr, "\Omega\Theta(\underline{X})"] & & & & \Omega W(\underline{X}) \arrow[dll, "\Omega\gamma(\underline{X})"'] \\
& & \Omega(\bigvee_{i=1}^{m} X_i) \arrow[d, "Q(\underline{X})"] & & \\
& & \Omega W(\underline{X}). & &
\end{tikzcd}
\tag{8}
$$

Here, $Q(\underline{X})$ is a left homotopy inverse of $\Omega\gamma(\underline{X})$ that exists by Lemma 6.7. The left and right quadrilaterals homotopy commute by the naturality of the Whitehead product and Theroem 6.1, as mentioned above. The upper triangle homotopy commutes by Proposition 6.6. Therefore the entire diagram homotopy commutes.

Since θ is a homotopy equivalence it has an inverse θ^{-1} which, by the homotopy commutativity of (6), satisfies $\Theta(\underline{\Sigma\Omega X}) \circ \theta^{-1} \simeq \gamma(\underline{\Sigma\Omega X})$. Define κ by the composite

$$\kappa\colon \Omega W(\underline{X}) \xrightarrow{\Omega\xi} \Omega W(\underline{\Sigma\Omega X}) \xrightarrow{\Omega\theta^{-1}} \Omega W(\underline{\Sigma\Omega X}) \xrightarrow{\Omega\zeta} \Omega W(\underline{X}).$$

As (8) homotopy commutes, we have $\Omega\Theta(\underline{X}) \circ \kappa \simeq \Omega\gamma(\underline{X}) \circ \Omega\zeta \circ \Omega\xi$. By Lemma 6.8, $\zeta \circ \xi$ is homotopic to the identity map on W. Therefore, $\Omega\Theta(\underline{X}) \circ \kappa \simeq \Omega\gamma(\underline{X})$. That is, κ is a lift of $\Omega\gamma(\underline{X})$ through $\Omega\Theta(\underline{X})$.

Further, composing with $Q(\underline{X})$ we obtain $Q(\underline{X}) \circ \Omega\Theta(\underline{X}) \circ \kappa \simeq Q(\underline{X}) \circ \Omega\gamma(\underline{X})$. By definition, $Q(\underline{X})$ is a left homotopy inverse of $\Omega\gamma(\underline{X})$. Therefore $Q(\underline{X}) \circ \Omega\Theta(\underline{X}) \circ \kappa$ is homotopic to the identity map on $\Omega W(\underline{X})$. That is, κ has a left homotopy inverse. We claim that this implies that κ is a homotopy equivalence. For, in homology, κ_* is a self-map of $H_*(\Omega W(\underline{X}))$ which has a left inverse. Since $H_*(\Omega W(\underline{X}))$ is a finite type module, this can occur only if κ_* is an isomorphism, and hence κ is a homotopy equivalence.

At this point, from $\Omega\Theta(\underline{X})\circ\kappa\simeq\Omega\gamma(\underline{X})$ we obtain $\Omega\Theta(\underline{X})\simeq\Omega\gamma(\underline{X})\circ\kappa^{-1}$. That is, there is a homotopy commutative diagram

$$\begin{tikzcd}
& \Omega W(\underline{X}) \arrow[dl, "\kappa^{-1}"'] \arrow[d, "\Omega\Theta(X)"] \\
\Omega W(\underline{X}) \arrow[r, "\Omega\gamma(X)"] & \Omega(\bigvee_{i=1}^m X_i).
\end{tikzcd}$$

It only remains to deloop this diagram. Since $W(\underline{X})$ is a suspension, write $W(\underline{X})=\Sigma W'(\underline{X})$. Consider the diagram

$$\begin{tikzcd}
\Sigma W'(\underline{X}) \arrow[r, "\Sigma E"] & \Sigma\Omega\Sigma W'(\underline{X}) \arrow[r, equal] \arrow[d, "\Sigma\kappa^{-1}"] & \Sigma\Omega\Sigma W'(\underline{X}) \arrow[r, "ev"] \arrow[d, "\Sigma\Omega\Theta(\underline{X})"] & \Sigma W'(\underline{X}) \arrow[d, "\Theta(\underline{X})"] \\
& \Sigma\Omega\Sigma W'(\underline{X}) \arrow[r, "\Sigma\Omega\gamma(\underline{X})"] \arrow[dr, "ev"] & \Sigma\Omega(\bigvee_{i=1}^m X_i) \arrow[r, "ev"] & \bigvee_{i=1}^m X_i. \\
& & \Sigma W'(\underline{X}) \arrow[ur, "\gamma(\underline{X})"'] &
\end{tikzcd}$$

The left upper square homotopy commutes by the previous diagram and the upper right square and lower triangle homotopy commute by the naturality of the evaluation map. The top row is homotopic to the identity map. So - going along the lower direction of the diagram - if $\theta=ev\circ\Sigma\kappa^{-1}\circ\Sigma E$ then $\Theta(\underline{X})\simeq\gamma(\underline{X})\circ\theta$, as required. $\square$

Lemma 6.10. *The lift θ in Proposition 6.9 is natural, in the sense that if there are maps $f_i\colon X_i\longrightarrow Y_i$ for $1\leq i\leq m$, then there is a homotopy commutative diagram*

$$\begin{tikzcd}
\bigvee_{\mathcal{I}}(k-1)(\Sigma X_{i_1}\wedge\cdots\wedge\Omega X_{i_k}) \arrow[r, "\psi"] \arrow[d, "\theta(\underline{X})"] & \bigvee_{\mathcal{I}}(k-1)(\Sigma X_{i_1}\wedge\cdots\wedge\Omega X_{i_k}) \arrow[d, "\theta(\underline{Y})"] \\
\bigvee_{\mathcal{I}}(k-1)(\Sigma X_{i_1}\wedge\cdots\wedge\Omega X_{i_k}) \arrow[r, "\psi"] & \bigvee_{\mathcal{I}}(k-1)(\Sigma X_{i_1}\wedge\cdots\wedge\Omega X_{i_k})
\end{tikzcd}$$

where ψ is the wedge sum of the maps $\Sigma f_{i_1}\wedge\cdots\wedge f_{i_k}$.

Proof. To compress notation, as in Proposition 6.9, let $W(\underline{X})=\bigvee_{\mathcal{I}}(k-1)(\Sigma X_{i_1}\wedge\cdots\wedge\Omega X_{i_k})$. Since $W(\underline{X})$ is a suspension there is a group structure on $[W(\underline{X}),W(\underline{X})]$ so we may take the difference $D=\psi\lambda(\underline{X})-\lambda(\underline{Y})\circ\psi$. To analyze the difference, consider the diagram

$$\begin{tikzcd}
W(\underline{X}) \arrow[rrr, "\psi"] \arrow[dr, "\Theta(\underline{X})"] \arrow[dd, "\theta(\underline{X})"'] & & & W(\underline{Y}) \arrow[dl, "\Theta(\underline{Y})"'] \arrow[dd, "\theta(\underline{Y})"] \\
& \bigvee_{i=1}^m X_i \arrow[r, "\bigvee_{i=1}^m f_i"] & \bigvee_{i=1}^m Y_i & \\
W(\underline{X}) \arrow[rrr, "\psi"] \arrow[ur, "\gamma(\underline{X})"] & & & W(\underline{Y}). \arrow[ul, "\gamma(\underline{Y})"']
\end{tikzcd}$$

The upper quadrilateral homotopy commutes by the naturality of the Whitehead product and the lower quadrilateral homotopy commutes by the naturality of Theorem 6.1. The two triangles homotopy commute by Proposition 6.9. The homotopy commutativity of the diagram as a whole implies that $\gamma(\underline{Y}) \circ \psi \circ \theta(\underline{X}) \simeq \gamma(\underline{Y}) \circ \theta(\underline{Y}) \circ \psi$. Thus $\gamma(\underline{Y}) \circ D$ is null homotopic. Therefore D lifts to the homotopy fibre of $\gamma(\underline{X})$. But Lemma 6.7 implies that there is a homotopy fibration sequence $\prod_{i=1}^{m} \Omega Y_i \xrightarrow{*} W(\underline{Y}) \xrightarrow{\gamma(\underline{Y})} \bigvee_{i=1}^{m} Y_i \longrightarrow \prod_{i=1}^{m} Y_i$. Thus D lifts through the trivial map and so is null homotopic. Hence $\psi \circ \theta(\underline{X}) \simeq \theta(\underline{Y}) \circ \psi$, as asserted. $\square$

Finally, the proof of Theorem 6.2 can be completed.

Proof of Theorem 6.2. The homotopy equivalence for $\Gamma(\underline{X})$ and the description of $\gamma(\underline{X})$ in terms of Whitehead products follows immediately from the definition of $\Theta(\underline{X})$, the lift in (6) and Proposition 6.9. The naturality statement holds by Lemma 6.10. $\square$

7. Whitehead products and Ganea's Theorem

Consider the following two maps: the inclusion $i\colon X \vee Y \longrightarrow X \times Y$ of the wedge into the product and the pinch map $q\colon X \vee Y \longrightarrow Y$ onto the right wedge summand. Observe that q is also the composite $X \vee Y \xrightarrow{i} X \times Y \xrightarrow{\pi_2} Y$, where π_2 is the projection onto the second factor. Define spaces F and G, and maps f and g, by the homotopy pullback diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d, "g"] & X \arrow[d, "i_1"] \\
F \arrow[r, "f"] & X \vee Y \arrow[r, "i"] \arrow[d, "q"] & X \times Y \arrow[d, "\pi_2"] \\
 & Y \arrow[r, equal] & Y
\end{tikzcd}$$

where i_1 is the inclusion of the first factor. As mentioned in Section 6, Ganea [G2] identified the homotopy type of F as $\Sigma\Omega X \wedge \Omega Y$ and the homotopy classes of f as a Whitehead product. The homotopy type of G is well known to be $X \rtimes \Omega Y$ but the homotopy class of g is not readily identifiable in terms of otherwise known maps. When X is a suspension then $X \rtimes \Omega Y \simeq X \vee (X \wedge \Omega Y)$, in which case the homotopy class of g should be identifiable. As the author can find no reference for this, we give a proof. There is no claim of anything new here, as the methods and results go back to Ganea [G2] and he surely knew everything stated in this section.

Let $i_X : X \longrightarrow X \vee Y$ and $i_Y : Y \longrightarrow X \vee Y$ be the inclusions of the respective wedge summands. Define ev_X and ev_Y by the composites $ev_X : \Sigma\Omega X \xrightarrow{ev} X \xrightarrow{i_X} X \vee Y$ and $ev_Y : \Sigma\Omega Y \xrightarrow{ev} Y \xrightarrow{i_Y} X \vee Y$.

Theorem 7.1. *The following hold:*

(a) *There is a homotopy fibration*

$$\Sigma\Omega X \wedge \Omega Y \xrightarrow{[ev_X, ev_Y]} X \vee Y \xrightarrow{\ \ i\ \ } X \times Y;$$

(b) *there is a homotopy fibration*

$$X \rtimes \Omega Y \xrightarrow{g} X \vee Y \xrightarrow{q} Y$$

where the restriction of g to X is i_X;

(c) *if $X = \Sigma X'$, then there is a choice of a homotopy equivalence $\Sigma X' \rtimes \Omega Y \simeq \Sigma X' \vee (\Sigma X' \wedge \Omega Y)$ such that the restriction of g to $\Sigma X' \wedge \Omega Y$ is $[i_X, ev_Y]$.*

Proof. Ganea [G2, 5.1] proved part (a). A proof of part (b) can be found in [Se, 7.7.7]. The ideas behind both which lead to part (c) are as follows. In general, for a space Z, let PZ be the path space of Z, where paths start at the basepoint of Z at time $t = 0$. Let $ev_1 \colon PZ \longrightarrow Z$ be the map which evaluates a path at time $t = 1$. The homotopy fibre of a map $h \colon A \longrightarrow Z$ is homotopy equivalent to the topological pullback of h and ev_1.

In our case, let Q be the topological pullback of the maps $X \vee Y \xrightarrow{i} X \times Y$ and $PX \times PY \xrightarrow{ev_1 \times ev_1} X \times Y$. Notice that the part of Q sitting over $* \vee Y$ is $\Omega X \times PY$, the part sitting over $X \vee *$ is $PX \times \Omega Y$, and the part sitting over $* \vee *$ is $\Omega X \times \Omega Y$. Thus $Q = \Omega X \times PY \cup_{\Omega X \times \Omega Y} PX \times \Omega Y$. Let $C\Omega X$ be the reduced cone on ΩX, where the cone point is at $t = 1$. It is well known that there is a homotopy equivalence $\psi_X \colon C\Omega X \longrightarrow PX$ given by $\psi_X(t, \omega) = \omega_t$, where ω_t is the path defined by $\omega_t(s) = \omega((1-t)s)$. Note that $\omega_0 = \omega$ and ω_1 is the constant path to the basepoint of X. Thus $Q \simeq \Omega X \times C\Omega Y \cup_{\Omega X \times \Omega Y} C\Omega X \times \Omega Y$, and the right side is the definition of the join $\Omega X * \Omega Y$. Ganea shows that the composite $\Sigma\Omega X \wedge \Omega Y \simeq \Omega X * \Omega Y \xrightarrow{\simeq} Q \longrightarrow X \vee Y$ is the Whitehead product $[ev_X, ev_Y]$, proving part (a).

Next, let P be the topological pullback of the maps $X \vee Y \xrightarrow{q} Y$ and $PY \xrightarrow{ev_1} Y$. Notice that the part of P sitting over $* \vee Y$ is $* \times PY$, the part sitting over $X \vee *$ is $X \times \Omega Y$, and the part sitting over $* \vee *$ is $* \times \Omega Y$. Thus $P = * \times PY \cup_{* \times \Omega Y} X \times \Omega Y$. The composite $X \longrightarrow P \longrightarrow X \vee Y$ is exactly i_X and contracting PY we obtain $P \simeq X \rtimes \Omega Y$. This proves part (b).

For part (c), observe that the projection $X \times Y \xrightarrow{\pi_2} Y$ induces a projection $PX \times PY \xrightarrow{\pi_2} PY$, so there is an induced map of topological pullbacks $Q \longrightarrow P$. In terms of the identifications for Q and P above, this map of pullbacks is

$$\Omega X \times PY \cup_{\Omega X \times \Omega Y} PX \times \Omega Y \longrightarrow * \times PY \cup_{* \times \Omega Y} X \times \Omega Y$$

where $PX \times PY \xrightarrow{ev_1 \times 1} X \times PY$ has been restricted to the subspaces $\Omega X \times PY$, $PX \times \Omega Y$ and $\Omega X \times \Omega Y$. Notice that the composite $C\Omega X \xrightarrow{\psi_X} PX \xrightarrow{ev_1} X$ sends (t, ω) to $\omega_t(1) = \omega(1-t)$. That is, $ev_1 \circ \psi_X = -ev$, where $\Sigma\Omega X \xrightarrow{ev} X$ is the canonical evaluation map. Thus the composite $\Sigma\Omega X \wedge \Omega Y \simeq \Omega X * \Omega Y \xrightarrow{\simeq} Q \longrightarrow P \simeq X \rtimes \Omega Y \longrightarrow X \wedge \Omega Y$ is $-ev \wedge 1$. Consequently, if $X = \Sigma X'$ and s is the composite $s \colon \Sigma X' \wedge \Omega Y \xrightarrow{\Sigma E \wedge 1} \Sigma\Omega\Sigma X' \wedge \Omega Y \xrightarrow{\simeq} Q \longrightarrow P \simeq X \rtimes \Omega Y$, then $\Sigma X' \wedge \Omega Y \xrightarrow{\gamma} \Sigma X' \rtimes \Omega Y \longrightarrow \Sigma X' \wedge \Omega Y$ is homotopic to $-id$, where id is the identity map. Thus s

is a section for the right map in the cofibration $\Sigma X' \xrightarrow{j} \Sigma X' \rtimes \Omega Y \longrightarrow \Sigma X' \wedge \Omega Y$, where j is the inclusion. Therefore $\Sigma X' \vee (\Sigma X' \wedge \Omega Y) \xrightarrow{j+s} X \rtimes \Omega Y$ is a homotopy equivalence.

Finally, from the pullback map $Q \longrightarrow P$ and the definition of s there is a homotopy commutative diagram

$$\begin{tikzcd}
\Sigma X' \wedge \Omega Y \arrow[r, "\Sigma E \wedge 1"] \arrow[drr, "s"'] & \Sigma\Omega\Sigma X' \wedge \Omega Y \arrow[r, "\simeq"] & Q \arrow[r] \arrow[d] & X \vee Y \arrow[d, equal] \\
 & & P \arrow[r] & X \vee Y.
\end{tikzcd}$$

By part (a) the map $\Sigma\Omega\Sigma X' \wedge \Omega Y \simeq Q \longrightarrow X \vee Y$ is homotopic to $[ev_X, ev_Y]$. Therefore the top row is homotopic to $[i_X, ev_Y]$. Hence, choosing the homotopy equivalence for $P \simeq \Sigma X' \rtimes \Omega Y$ determined by $j+s$, the restriction of $\Sigma X. \rtimes \Omega Y \longrightarrow X \vee Y$ to $\Sigma X' \wedge \Omega Y$ is homotopic to $[i_X, ev_Y]$. $\square$

8. A refinement of Theorem 6.2 in the case of suspensions

Return to the homotopy fibration $\Gamma(\underline{X}) \xrightarrow{\gamma(\underline{X})} \bigvee_{i=1}^m X_i \longrightarrow \prod_{i=1}^m X_i$. In Theorem 6.2 a homotopy equivalence was given for $\Gamma(\underline{X})$ in terms of a wedge sum of spaces $\Sigma\Omega X_{i_1} \wedge \cdots \Omega X_{i_k}$ that let one identify the homotopy class of $\gamma(\underline{X})$ as a wedge sum of iterated Whitehead products formed from the maps t_j. When each $X_i = \Sigma Y_i$ then the James decomposition of $\Sigma\Omega\Sigma Y$ may be iteratively applied to the wedge summands of $\Gamma(\underline{\Sigma Y})$. It will be useful later to show that in this case the map $\gamma(\underline{\Sigma Y})$ can be described in terms of iterated Whitehead products formed from the maps s_j that include wedge summands into $\bigvee_{i=1}^m \Sigma Y_i$.

Let L be the free Lie algebra $L = L\langle v_1, \ldots, v_m\rangle$. Recall the construction in Section 6 of a sequence of free Lie algebras $L_m \subset L_{m-1} \subset \cdots \subset L_1 \subset L$ with $L_m \cong [L, L]$ and a Lie basis W_m of L_m was identified recursively in terms of the Lie bases W_i of L_i. There is a way of topologically enumerating the Lie basis W_m.

Pinching $\bigvee_{i=1}^m \Sigma Y_i$ to ΣY_1, we obtain a homotopy fibration

$$D_1 \longrightarrow \bigvee_{i=1}^m \Sigma Y_i \longrightarrow \Sigma Y_1.$$

By Theorem 4.1, there are homotopy equivalences

$$D_1 \simeq (\bigvee_{i=2}^m \Sigma Y_i) \rtimes \Omega\Sigma Y_1 \simeq (\bigvee_{i=2}^m \Sigma Y_i) \vee (\bigvee_{i=2}^m (\Sigma Y_i \wedge \Omega\Sigma Y_1)).$$

Applying Lemma 4.2 to each $\Sigma Y_i \wedge \Omega\Sigma Y_1$, we obtain a homotopy equivalence

$$D_1 \simeq (\bigvee_{i=2}^m \Sigma Y_i) \vee (\bigvee_{i=2}^m \bigvee_{k_1=1}^\infty \Sigma Y_i \wedge (Y_1)^{(k_1)}).$$

Observe that there is a one-to-one correspondence between the wedge summands of D_1 and the elements in the Lie basis $W_1 = \{ad^{k_1}(x)(v_1) \mid x \in \{v_2, \ldots, v_m\}, k_1 \geq 0\}$ for L_1. Regarding $Y_i \wedge (Y_1)^{(0)}$

as Y_i, the wedge decomposition for D_1 can be reorganized as

$$D_1 \simeq \bigvee_{i=2}^{m} \bigvee_{k_1=0}^{\infty} \Sigma Y_i \wedge (Y_1)^{(k_1)}. \tag{9}$$

Iterating, assume that D_t has been defined, there is a homotopy equivalence

$$D_{t-1} \simeq \bigvee_{i=t}^{m} \bigvee_{k_1,\ldots,k_{t-1}=0}^{\infty} \Sigma Y_i \wedge (Y_1)^{(k_1)} \wedge \cdots \wedge (Y_t)^{(k_{t-1})} \tag{10}$$

and the wedge summands of D_{t-1} are in one-to-one correspondence with the Lie basis W_{t-1} of L_{t-1}. Note that ΣY_i is a wedge summand for D_{t-1} for all $i \geq t$ - in (10) they are the wedge summands indexed by taking $k_1 = \ldots = k_{t-1} = 0$. Write D_{t-1} as $\Sigma Y_t \vee \overline{D}_{t-1}$ and define D_t by the homotopy fibration

$$D_t \longrightarrow D_{t-1} \xrightarrow{q_t} \Sigma Y_t$$

where q_t is the pinch map. Note that this corresponds to writing W_{t-1} as $\{v_t\} \cup \overline{W}_{t-1}$, and defining L_t as the kernel of the Lie algebra map $L_{t-1} \longrightarrow L_{ab}\langle v_t \rangle$. Since D_{t-1} is a suspension the same is true for $\overline{D}_{t-1}$, so by Theorem 4.1 there are homotopy equivalences

$$D_t \simeq \overline{D}_{t-1} \rtimes \Omega\Sigma Y_t \simeq \overline{D}_{t-1} \vee (\overline{D}_{t-1} \wedge \Omega\Sigma Y_t).$$

Since each wedge summand of $\overline{D}_{t-1}$ is a suspension Lemma 4.2 may be applied to $\overline{D}_{t-1} \wedge \Omega\Sigma Y_t$ in order to obtain a homotopy equivalence

$$D_t \simeq \overline{D}_{t-1} \vee \big(\bigvee_{k_t=1}^{\infty} \overline{D}_{t-1} \wedge (Y_t)^{(k_t)} \big).$$

Observe that the wedge summands of D_t are in one-to-one correspondence with the elements of the Lie basis W_t of L_t. Using $k_t = 0$ to denote $\overline{D}_{t-1}$ and substituting in the decomposition for $\overline{D}_{t-1}$, we obtain

$$D_t \simeq \bigvee_{k_t=0}^{\infty} \overline{D}_{t-1} \wedge (Y_t)^{(k_t)} \simeq \bigvee_{i=t+1}^{m} \bigvee_{k_1,\ldots,k_t=0}^{\infty} \Sigma Y_i \wedge (Y_1)^{(k_1)} \wedge \cdots \wedge (Y_t)^{(k_t)}.$$

At the end of the iteration we obtain a space D_m and a homotopy equivalence

$$D_m \simeq \bigvee_{k_1,\ldots,k_m=0}^{\infty} (Y_1)^{(k_1)} \wedge \cdots \wedge (Y_t)^{(k_m)}$$

where the index set is in one-to-one correspondence with the Lie basis W_m of L_m. The topological analogue of the identification of L_m with $[L, L]$ in Lemma 6.3 is the following.

Lemma 8.1. *There is a homotopy equivalence $D_m \cong \Gamma(\underline{\Sigma Y})$.*

Proof. Observe that there is a homotopy fibration diagram

$$\begin{tikzcd}
D_1 \arrow[r] \arrow[d, "q_2"] & \bigvee_{i=1}^{m} \Sigma Y_i \arrow[r, "q_1"] \arrow[d] & \Sigma Y_1 \arrow[d, equal] \\
\Sigma Y_2 \arrow[r] & \Sigma Y_1 \times \Sigma Y_2 \arrow[r] & \Sigma Y_1.
\end{tikzcd}$$

The left square is therefore a homotopy pullback, so the homotopy fibre of $\bigvee_{i=1}^m \Sigma Y_i \longrightarrow \Sigma Y_1 \times \Sigma Y_2$ is the same as that of q_2, which is D_2. We now argue by induction. Assume that D_t is the homotopy fibre of the map $\bigvee_{i=1}^m \Sigma Y_i \longrightarrow \prod_{i=1}^t \Sigma Y_i$. Then there is a homotopy fibration diagram

$$\begin{tikzcd}
D_t \arrow[r] \arrow[d, "q_{t+1}"] & \bigvee_{i=1}^m \Sigma Y_i \arrow[r] \arrow[d] & \prod_{i=1}^t \Sigma Y_i \arrow[d, equal] \\
\Sigma Y_{t+1} \arrow[r] & \prod_{i=1}^{t+1} \Sigma Y_i \arrow[r] & \prod_{i=1}^t \Sigma Y_i.
\end{tikzcd}$$

Therefore the homotopy fibre of $\bigvee_{i=1}^m \Sigma Y_i \longrightarrow \prod_{i=1}^{t+1} \Sigma Y_i$ is the same as that of q_{t+1}, which is D_{t+1}. Hence, by induction, there is a homotopy fibration

$$D_m \longrightarrow \bigvee_{i=1}^m \Sigma Y_i \longrightarrow \prod_{i=1}^m \Sigma Y_i.$$

Observe that the map on the left is the inclusion of the wedge into the product, and by definition the homotopy fibre of this inclusion is $\Gamma(\underline{\Sigma Y})$. Therefore $D_m \simeq \Gamma(\underline{\Sigma Y})$. $\square$

The correspondence between the wedge summands of D_m and the Lie basis elements of W_m has a geometric interpretation. Recall that for $1 \leq j \leq m$ the map $s_j \colon \Sigma Y_j \longrightarrow \bigvee_{i=1}^m \Sigma Y_i$ is the inclusion of the j^{th}-wedge summand. Suppose that the Lie basis element $w \in W_m$ corresponds to the wedge summand $\Sigma Y_{i_1}^{(k_{i_1})} \wedge \cdots \wedge Y_{i_\ell}^{(k_{i_\ell})}$ of D_m. Define

$$\Phi_w(\underline{\Sigma Y}) \colon \Sigma Y_{i_1}^{(k_{i_1})} \wedge \cdots \wedge Y_{i_\ell}^{(k_{i_\ell})} \longrightarrow \bigvee_{i=1}^m \Sigma Y_i$$

as the iterated Whitehead product in the maps s_j corresponding to w. Note here that the total bracket length is $k_{i_1} + \cdots + k_{i_\ell}$ and the entries s_{j_t} in the Samelson products may be repeated. Let

$$\Phi(\underline{\Sigma Y}) \colon D_m \longrightarrow \bigvee_{i=1}^m \Sigma Y_i$$

be the wedge sum of the maps Φ_w for all $w \in W_m$.

Proposition 8.2. *There is a homotopy commutative diagram*

$$\begin{tikzcd}
 & D_m \arrow[dl, "\phi"'] \arrow[d, "\Phi(\underline{\Sigma Y})"] \\
\Gamma(\underline{\Sigma Y}) \arrow[r, "\gamma(\underline{\Sigma Y})"] & \bigvee_{i=1}^m \Sigma Y_i
\end{tikzcd}$$

where ϕ is a homotopy equivalence. Further, ϕ is natural for maps $Y_i \longrightarrow Z_i$.

Proof. Consider the homotopy fibration $\Gamma(\underline{\Sigma Y}) \xrightarrow{\gamma(\underline{\Sigma Y})} \bigvee_{i=1}^m \Sigma Y_i \longrightarrow \prod_{i=1}^m \Sigma Y_i$. Every Whitehead product in the wedge sum defining $\Phi(\underline{\Sigma Y})$ involves at least two distinct s_j's since the corresponding Lie basis element of W_m involves two distinct v_j's. Any such Whitehead product vanishes when composed into $\prod_{i=1}^m \Sigma Y_i$. Therefore the map $\Phi(\underline{\Sigma Y})$ lifts through $\gamma(\underline{X})$ to a map $D_m \xrightarrow{\phi} \Gamma(\underline{X})$ which makes the diagram in the statement of the theorem homotopy commute.

Let

$$s_w(\underline{\Sigma Y})\colon Y_{i_1}^{(k_{i_1})}\wedge\cdots\wedge Y_{i_\ell}^{(k_{i_\ell})}\longrightarrow\Omega(\bigvee_{i=1}^m\Sigma Y_i)$$

be the adjoint of $\Phi_w(\underline{\Sigma Y})$. Noting that D_m is a suspension, write $D_m=\Sigma D'_m$, and let

$$s(\underline{\Sigma Y})\colon D'_m\longrightarrow\Omega(\bigvee_{i=1}^m\Sigma Y_i)$$

be the adjoint of $\Phi(\underline{\Sigma Y})$. The argument from here on is similar to that in the proof for Proposition 6.6. Take homology with field coefficients. Let $x_{i_t}\in\widetilde{H}_*(Y_{i_t})$ and write $w(x_{i_1},\ldots,x_{i_\ell})$ for the bracket in $L\langle V_{i_1}\rangle\amalg\cdots\amalg L\langle v_{i_t}\rangle$ in which each v_{i_t} is replaced by x_{i_t}. Note that the x_{i_t}'s may be repeated. Observe that

$$\mathrm{Im}(s_w(\underline{\Sigma Y}))_*=\{w(x_{i_1},\ldots,x_{i_\ell})\mid x_{i_t}\in\widetilde{H}_*(Y_{i_t})\}\subset L\langle V_{i_1}\rangle\amalg\cdots\amalg L\langle V_{i_t}\rangle$$

and that $(s_w(\underline{\Sigma Y}))_*$ is an injection. Next, if $w_1,w_2\in W_m$ are distinct Lie basis elements then the images of $(s_{w_1}(\underline{\Sigma Y}))_*$ and $(s_{w_2}(\underline{\Sigma Y}))_*$ are linearly independent because they correspond to distinct bracketings, and therefore the map $[L\langle V\rangle,L\langle V\rangle]\xrightarrow{s(\underline{\Sigma Y})_*}UL\langle V\rangle$ is an injection. Applying the James construction to multiplicatively extend $s(\underline{\Sigma Y})$ to $\Omega\Phi(\underline{\Sigma Y})$ therefore implies that $(\Omega\Phi(\underline{\Sigma Y}))_*$ is an injection. The homotopy commutativity of the diagram in the statement of the theorem therefore implies that $(\Omega\phi)_*$ is an injection. Now deduce as in Proposition 6.6 that $\Omega\phi$ is a homotopy equivalence, and therefore that ϕ is a homotopy equivalence. The naturality of ϕ follows as for the proof of Proposition 6.10. $\square$

Proposition 8.2 immediately implies the following.

Theorem 8.3. *Let $Y_1,\ldots,Y_m$ be pointed, path-connected CW-complexes. There is a homotopy fibration*

$$D_m\xrightarrow{\Phi(\underline{\Sigma Y})}\bigvee_{i=1}^m\Sigma Y_i\longrightarrow\prod_{i=1}^m\Sigma Y_i$$

which is natural for maps $Y_i\longrightarrow Z_i$. $\square$

There is an analogous refinement of Theorem 7.1. It was shown that there is a homotopy fibration $\Sigma X\rtimes\Omega Z\xrightarrow{g}X\vee Z\xrightarrow{q}Z$ where $\Sigma X\rtimes\Omega Z\simeq\Sigma X\vee(\Sigma X\wedge\Omega Z)$, the restriction of g to ΣX is the inclusion $i_{\Sigma X}$, and the restriction of g to $\Sigma X\wedge\Omega Z$ is the Whitehead product $[i_{\Sigma X},ev_Z]$. If $Z=\Sigma Y$ then by Lemma 4.2 there is a homotopy equivalence $\Sigma X\wedge\Omega\Sigma Y\simeq\bigvee_{n=1}^\infty\Sigma X\wedge Y^{(n)}$. Using the Lie algebra L_1 and its topological enumeration D_1 instead of L_m and D_m, and then arguing as for Proposition 8.2 and Theorem 8.3 we obtain the following.

Theorem 8.4. *Let X and Y be pointed, path-connected CW-complexes. There is a homotopy fibration*

$$\Sigma X\vee\left(\bigvee_{n=1}^\infty\Sigma X\wedge Y^{(n)}\right)\xrightarrow{\Psi}\Sigma X\vee\Sigma Y\xrightarrow{q}\Sigma Y$$

where the restriction of Ψ to ΣX is $i_{\Sigma X}$ and the restriction of Ψ to $\Sigma X \wedge Y^{(n)}$ is the iterated Whitehead product $ad^n(i_{\Sigma X})(i_{\Sigma Y})$. All of this is natural for maps $X \longrightarrow X'$ and $Y \longrightarrow Y'$. $\square$

9. Thin products and Whitehead products

Recall from the Introduction that if $X_1, \dots, X_m$ are pointed spaces then the thin product is defined by $P^m(\underline{X}) = (\underline{X}, \underline{*})^K_D$ where K is the simplicial complex consisting of m disjoint points. Noting that $(\underline{X}, \underline{*})^K \simeq \bigvee_{i=1}^m X_i$, from the map $(\underline{X}, \underline{*})^K \xrightarrow{\varphi} (\underline{X}, \underline{*})^K_D$ we obtain a homotopy fibration

$$F^m(\underline{X}) \xrightarrow{f^m(\underline{X})} \bigvee_{i=1}^m X_i \xrightarrow{\ \varphi\ } P^m(\underline{X}).$$

For $1 \leq j \leq m$, recall again that

$$s_j \colon X_j \longrightarrow \bigvee_{i=1}^m X_i$$

is the inclusion of the j^{th}-wedge summand. Suppose that for $1 \leq k \leq t$ there are maps $a_k \colon \Sigma Y_k \longrightarrow X_{j_k}$ where $1 \leq j_k \leq m$. Let b_k be the composite

$$b_k \colon \Sigma Y_k \xrightarrow{a_k} X_{j_k} \xrightarrow{s_{j_k}} \bigvee_{i=1}^m X_i.$$

Let

$$w \colon \Sigma Y_1 \wedge \cdots \wedge Y_t \longrightarrow \bigvee_{i=1}^m X_i$$

be an iterated Whitehead product formed from the maps a_k. We say that w has length t and involves the maps $\{s_{j_1}, \dots, s_{j_t}\}$.

Theorem 9.1. *Let w be a Whitehead product on $\bigvee_{i=1}^m X_i$ formed from the maps b_k. Suppose that w has length $t \geq m$ and involves all the maps s_j for $1 \leq j \leq m$. Then w lifts through $F^m(\underline{X}) \xrightarrow{f^m(\underline{X})} \bigvee_{i=1}^m X_i$.*

Proof. Recall that K is m disjoint points. For any proper subset $I \subset [m]$, by Lemma 2.3 the projection $X^m \longrightarrow X^I$ induces a map of polyhedral products $(\underline{X}, \underline{*})^K \longrightarrow (\underline{X}, \underline{*})^{K_I}$ which, in this case, is the equivalent to the pinch map $p_I \colon \bigvee_{i=1}^m X_i \longrightarrow \bigvee_{i \in I} X_i$. Observe that if $j \notin I$ then the composite $\Sigma X_j \xrightarrow{s_j} \bigvee_{i=1}^m X_i \xrightarrow{p_I} \bigvee_{i \in I} X_i$ is trivial, since s_j is the inclusion of X_j into the wedge. Since I is a proper subset of $[m]$, we can always find a j such that $p_I \circ s_j$ is null homotopic. As w is a Whitehead product involving all the maps s_j for $1 \leq j \leq m$, the naturality of the Whitehead product implies that $p_I \circ w$ is null homotopic. This holds for any proper subset I of $[m]$, so $p_I \circ w$ is null homotopic for all $I \subsetneq [m]$.

Let $\widetilde{w} \colon Y_1 \wedge \cdots \wedge Y_t \longrightarrow \Omega(\bigvee_{i=1}^m X_i)$ be the adjoint of w and consider the composite

$$Y_1 \wedge \cdots \wedge Y_m \xrightarrow{\widetilde{w}} \Omega(\bigvee_{i=1}^m X_i) \xrightarrow{\Omega\varphi} \Omega P^m(\underline{X}).$$

By Theorem 5.3, every factor in the homotopy decomposition of $\Omega P^m(\underline{X}) = \Omega(\underline{X}, \underline{*})^K_D$ is also a factor of $\Omega(\underline{X}, \underline{*})^{K_I}$ for some $I \subsetneq [m]$, and the decomposition is compatible with the maps Ωp_I.

Therefore as $p_I \circ w$ is null homotopic for all $I \subsetneq [m]$, so is $\Omega p_I \circ \widetilde{w}$. Thus $\Omega\varphi \circ \widetilde{w}$ is null homotopic, implying that its adjoint $\varphi \circ w$ is null homotopic. Hence w lifts through $f^m(\underline{X})$. $\square$

A special case is when each of the spaces X_i equals a common space X. Then we write $P^m(X)$ for the thin product, and have a homotopy fibration $F^m(X) \xrightarrow{f^m(X)} \bigvee_{i=1}^m X \longrightarrow P^m(X)$. In this case Theorem 9.1 refines. Consider the composite $F^m(X) \xrightarrow{f^m(X)} \bigvee_{i=1}^m X \xrightarrow{\nabla} X$ which is used to define the weak cocategory of X.

Lemma 9.2. *Let w be a Whitehead product on X of length $t \geq m$. Then w lifts through ∇ to a Whitehead product on $\bigvee_{i=1}^m X$ of length t involving all the inclusions s_j for $1 \leq j \leq m$.*

Proof. Suppose that w is a Whitehead product on X of length $t \geq m$, where w is formed from maps $a_k\colon \Sigma Y_k \longrightarrow X$ for $1 \leq k \leq t$. For $1 \leq k \leq m$, let b_j be the composite $\Sigma Y_j \xrightarrow{a_j} X \xrightarrow{s_j} \bigvee_{i=1}^m X$. If $t > m$, then for $m < k \leq t$, let b_k be the composite $\Sigma Y_k \xrightarrow{a_k} X \xrightarrow{s_m} \bigvee_{i=1}^m X$. Let $\overline{w}$ be the Whitehead product of the maps b_k for $1 \leq k \leq t$, where the bracketing order is the same as for w. Then the naturality of the Whitehead product implies that $w \simeq \nabla \circ \overline{w}$. Thus $\overline{w}$ lifts w through ∇, it has the same length as w, and involves all m inclusions s_j. $\square$

Combining the lifts in Lemma 9.2 and Theorem 9.1 we immediately obtain the following.

Theorem 9.3. *Any Whitehead product on X of length $t \geq m$ lifts through the composite $F^m(X) \xrightarrow{f^m(X)} \bigvee_{i=1}^m X \xrightarrow{\nabla} X$.* $\square$

Next, we aim towards Theorem 9.5 which is a sort of converse to Theorem 9.1. While Theorem 9.1 says that any Whitehead product of length $t \geq m$ involving all m maps s_j lifts through $f^m(\underline{X})$, Theorem 9.5 says that the homotopy class of $f^m(\underline{X})$ is completely determined by length $t \geq m$ Whitehead products involving all m maps s_j. To see this we modify the proof of Proposition 4.10 that identified the homotopy type of $F^m(\underline{X})$ in order to take into account the Whitehead product information in Theorems 6.2 and 7.1.

First, we require a general lemma.

Lemma 9.4. *Suppose that there are maps*

$$f = \bigvee_{i=1}^{s} f_i\colon \bigvee_{i=1}^{s} \Sigma A_i \longrightarrow Z \qquad g = \bigvee_{j=1}^{t} g_j\colon \bigvee_{j=1}^{t} \Sigma B_j \longrightarrow Z.$$

Then the Whitehead product $[f, g]$ is homotopic to the wedge sum of Whitehead products $\bigvee_{i=1}^s \bigvee_{j=1}^t [f_i, g_j]$.

Proof. Denoting the adjoint of a map u by $\tilde{u}$, it is equivalent to show that the Samelson product of $\langle \tilde{f}, \tilde{g}\rangle$ is homotopy equivalent to the wedge of Samelson products $\bigvee_{i=1}^s \bigvee_{j=1}^t \langle \tilde{f}_i, \tilde{g}_j\rangle$. But this is clear from the pointwise definition of the Samelson product of two maps u and v as $\langle u, v\rangle(x, y) = u(x)v(y)u(x)^{-1}v(y)^{-1}$. $\square$

Theorem 9.5. *Let $X_1, \ldots, X_m$ be a simply-connected, pointed CW-complexes. Then the homotopy fibration*

$$F^m(\underline{X}) \xrightarrow{f^m(\underline{X})} \bigvee_{i=1}^{m} X_i \longrightarrow P^m(\underline{X})$$

satisfies the following:

(a) $F^m(\underline{X}) \simeq \bigvee_{\alpha\in\mathcal{I}} \Sigma(\Omega X_1)^{(\alpha_1)} \wedge\cdots\wedge (\Omega X_m)^{(\alpha_m)}$ *where* $\alpha_i \geq 1$ *for each* $1 \leq i \leq m$*;*

(b) *the restriction of* $f^m(\underline{X})$ *to* $\Sigma(\Omega X_1)^{(\alpha_1)}\wedge\cdots\wedge(\Omega X_m)^{(\alpha_m)}$ *is an iterated Whitehead product of length* $t \geq m$ *formed from the maps* $t_j\colon \Sigma\Omega X_j \xrightarrow{ev} X_j \xrightarrow{s_j} \bigvee_{i=1}^m X_i$*, where each* t_j *for* $1 \leq j \leq m$ *appears at least once;*

(c) *parts (a) and (b) are natural for maps of spaces* $X_i \longrightarrow Y_i$*.*

Proof. The construction in Section 4 worked for any polyhedral product $(\underline{CX}, \underline{X})^K$ where K is a totally homology fillable simplicial complex. Specialize to the case of $(\underline{C\Omega X}, \underline{\Omega X})^K$ when K is m disjoint points. Then the homotopy fibration $(\underline{C\Omega X}, \underline{\Omega X})^K \longrightarrow (\underline{X}, \underline{*})^K \longrightarrow \prod_{i=1}^m X_i$ is a model for the homotopy fibration $\Gamma(\underline{X}) \xrightarrow{\gamma(\underline{X})} \bigvee_{i=1}^m X_i \longrightarrow \prod_{i=1}^m X_i$. Therefore, by Theorem 6.2 there is a homotopy decomposition of $(\underline{C\Omega X}, \underline{\Omega X})^K$ as a wedge sum of spaces $\Sigma\Omega X_{i_1} \wedge\cdots\wedge \Omega X_{i_k}$ which can be chosen so that the restriction of the map $(\underline{C\Omega X}, \underline{\Omega X})^K \xrightarrow{\gamma(\underline{X})} (\underline{X}, \underline{*})^K \simeq \bigvee_{i=1}^m X_i$ to a wedge summand $\Sigma\Omega X_{i_1} \wedge\cdots\wedge \Omega X_{i_t}$ is an iterated Whitehead product of the maps t_j.

With $(\underline{C\Omega X}, \underline{\Omega X})^K \longrightarrow \bigvee_{i=1}^m X_i$ as the starting point in this case, the first step in the construction in Section 4 was to divide the wedge summands of $(\underline{C\Omega X}, \underline{\Omega X})^K$ so that $(\underline{C\Omega X}, \underline{\Omega X})^K \simeq \Sigma B_1 \vee \Sigma C_1$, where ΣB_1 consists of those wedge summands having ΩX_1 as a smash factor and ΣC_1 consists of those wedge summands not having ΩX_1 as a smash factor. Pinching to ΣC_1, we obtain a homotopy fibration $\Sigma B_1 \rtimes \Omega\Sigma C_1 \xrightarrow{g_1} \Sigma B_1 \vee \Sigma C_1 \longrightarrow \Sigma C_1$. By Theorem 8.4, there is a homotopy equivalence

$$\Sigma B_1 \rtimes \Omega\Sigma C_1 \simeq \bigvee_{n=1}^{\infty} (\Sigma B_1 \vee (C_1)^{(n)})$$

which may be chosen so that the restriction of g_1 to ΣB_1 is i_L and the restriction to $\Sigma B_1 \wedge (C_1)^{(n)}$ is an iterated Whitehead product of the maps i_L and i_R where i_L appears once and i_R appears n times. Regarding ΣB_1 and ΣC_1 as a wedge of spaces of the form $\Sigma\Omega X_{i_1} \wedge\cdots\wedge \Omega X_{i_k}$, by Lemma 9.4 each iterated Whitehead product of the maps i_L and i_R is homotopic to a wedge sum of iterated Whitehead products of the inclusion maps of the summands $\Sigma\Omega X_{i_1} \wedge\cdots\wedge \Omega X_{i_k}$ into $(\underline{C\Omega X}, \underline{\Omega X})^K$. Therefore, as $(\underline{C\Omega X}, \underline{\Omega X})^K \xrightarrow{\gamma(\underline{X})} \bigvee_{i=1}^m X_i$ is a wedge sum of iterated Whitehead products of the maps t_j, the naturality of the Whitehead product implies that the composite $\Sigma B_1 \rtimes \Omega\Sigma C_1 \xrightarrow{g_1} \Sigma B_1 \vee \Sigma C_1 \simeq (\underline{C\Omega X}, \underline{\Omega X})^K \xrightarrow{\gamma(\underline{X})} \bigvee_{i=1}^m X_i$ is homotopic to a wedge sum of iterated Whitehead products (of iterated Whitehead products) of the maps t_j. Further, as each wedge summand of $\Sigma B_1 \rtimes \Omega\Sigma C_1$ has ΩX_1 as a smash factor, each of the Whitehead products has t_1 appearing at least once.

The next step in the construction identifying the homotopy type of $F^m(\underline{X})$ was to divide $G_1 \simeq \Sigma B_1 \rtimes \Omega\Sigma C_1$ into a wedge $\Sigma B_2 \vee \Sigma C_2$ where each wedge summand of ΣB_2 has both ΩX_1 and ΩX_2 as smash factors while the wedge summands of ΣC_2 have ΩX_1 as a smash factor but not ΩC_2. Then one pinches G_1 to ΣC_2 and calls the homotopy fibre G_2. This process is iterated until G_m is identified as being homotopy equivalent to $F^m(\underline{X})$. At each step in the iteration we may argue as in the previous paragraph to identify the composite $G_i \longrightarrow G_{i-1} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow (\underline{C\Omega X}, \underline{\Omega X})^K \longrightarrow \bigvee_{i=1}^m X_i$ as homotopic to a wedge sum of iterated Whitehead products of the maps t_j. As each of $\Omega X_1, \ldots, \Omega X_m$ appears as a smash factor in each wedge summand of G_m, each of the corresponding Whitehead products has $t_1, \ldots, t_m$ appearing at least once. This proves parts (a) and (b).

The naturality statement in part (c) follows from the naturality of Theorems 6.2 and 7.1. $\square$

When each X_i equals a common space X Theorem 9.5 implies the following.

Corollary 9.6. *Let X be a simply-connected, pointed CW-complex. Then the homotopy fibration*
$$F^m(X) \xrightarrow{f^m(X)} \bigvee_{i=1}^m X \longrightarrow P^m(X)$$
satisfies the following:

(a) $F^m(X) \simeq \bigvee_{\alpha\in\mathcal{I}} \Sigma(\Omega X)^{(t_\alpha)}$;

(b) *the restriction of $f^m(X)$ to $\Sigma(\Omega X)^{(t_\alpha)}$ is an iterated Whitehead product of length $t \geq m$ formed from the maps $t_j\colon \Sigma\Omega X \xrightarrow{ev} X \xrightarrow{s_j} \bigvee_{i=1}^m X$ where each t_j for $1 \leq j \leq m$ appears at least once;*

(c) *parts (a) and (b) are natural for a map of spaces $X \longrightarrow Y$.*

$\square$

We also need a refined version of Theorem 9.5 in the case when each X_i is a suspension.

Theorem 9.7. *Let $Y_1, \ldots, Y_m$ be a simply-connected, pointed CW-complexes. Then the homotopy fibration*
$$F^m(\underline{\Sigma Y}) \xrightarrow{f^m(\underline{\Sigma Y})} \bigvee_{i=1}^m \Sigma Y_i \longrightarrow P^m(\underline{\Sigma Y})$$
satisfies the following:

(a) $F^m(\underline{\Sigma Y}) \simeq \bigvee_{\beta\in\mathcal{J}} \Sigma(Y_1)^{(\beta_1)} \wedge \cdots \wedge (Y_m)^{(\beta_m)}$ *where $\beta_i \geq 1$ for each $1 \leq i \leq m$;*

(b) *the restriction of $f^m(\underline{\Sigma Y})$ to $\Sigma(Y_1)^{(\beta_1)} \wedge \cdots \wedge (Y_m)^{(\beta_m)}$ is an iterated Whitehead product of length $t \geq m$ formed from the maps $\Sigma Y_j \xrightarrow{s_j} \bigvee_{i=1}^m \Sigma Y_i$, where each s_j for $1 \leq j \leq m$ appears at least once;*

(c) *parts (a) and (b) are natural for maps of spaces $Y_i \longrightarrow Z_i$.*

Proof. Argue exactly as for Theorem 9.5 using Theorem 8.3 as the starting point rather than Theorem 6.2, and organizing the B_i, C_i wedge summands so that B_i contains all the smash products

having $Y_1, \dots, Y_i$ as factors (rather than $\Omega\Sigma Y_1, \dots, \Omega\Sigma Y_i$) and C_i contains all the smash products having $Y_1, \dots, Y_{i-1}$ as factors but not Y_i. $\square$

Part 3. Cocategory and Nilpotence.

10. Cocategory and Nilpotence I

In this section the main result of the paper, Theorem 1.6, is proved: if X is simply-connected then $\mathrm{wcocat}(X) = m$ if and only if $\mathrm{nil}(\Omega X) = m$. Before starting we record a simple lemma.

Lemma 10.1. *Suppose that a and b are positive integers satisfying: (i) $a = m$ implies $b \leq m$, and (ii) $b = m$ implies $a \leq m$. Then $a = m$ if and only if $b = m$.*

Proof. If $a = m$ then by (i), $b \leq m$. So either $b = m$ or $b < m$. But if $b < m$ then by (ii), $a < m$, a contradiction. Therefore $b = m$. Similarly, if $b = m$ then $a = m$. $\square$

Proof of Theorem 1.6. We will use the homotopy fibration

$$F^{m+1}(X) \xrightarrow{f^{m+1}(X)} \bigvee_{i=1}^{m+1} X \longrightarrow P^{m+1}(X) \tag{11}$$

from Corollary 9.6.

Suppose that $\mathrm{wcocat}(X) = m$. By the definition of weak cocategory, this implies that the composition $F^{m+1}(X) \xrightarrow{f^{m+1}(X)} \bigvee_{i=1}^{m+1} X \xrightarrow{\nabla} X$ is null homotopic. Consider the Samelson product $(\Omega X)^{(m+1)} \xrightarrow{c_m} \Omega X$, where c_m is the $(m+1)$-fold Samelson product of the identity map on ΩX with itself. Let $w_m \colon \Sigma(\Omega X)^{(m+1)} \longrightarrow X$ be the adjoint of c_m. Then w_m is the $(m+1)$-fold Whitehead product of the evaluation map $\Sigma\Omega X \xrightarrow{\overline{ev}} X$ with itself. By Theorem 9.3, w_m factors through $\nabla \circ f^{m+1}(X)$. But by hypothesis, $\nabla \circ f^{m+1}(X)$ is null homotopic. Therefore w_m is null homotopic and hence its adjoint c_m is null homotopic. This implies that $\mathrm{nil}(G) \leq m$.

Conversely, suppose that $\mathrm{nil}(G) = m$. By Corollary 9.6, $F^{m+1}(X) \simeq \bigvee_{\alpha \in \mathcal{I}} \Sigma(\Omega X)^{(t_\alpha)}$ and the restriction of $f^{m+1}(X)$ to each wedge summand is an iterated Whitehead product of length $\geq m+1$. Therefore $\nabla \circ f^{m+1}(X)$ maps each wedge summand $\Sigma(\Omega X)^{(t_\alpha)}$ to X by an iterated Whitehead product of length $\geq m+1$. The adjoint $(\Omega X)^{(t_\alpha)} \longrightarrow \Omega X$ is a Samelson product of length $\geq m+1$, and so factors through c_m. By hypothesis, c_m is null homotopic. Therefore, so is $(\Omega X)^{(t_\alpha)} \longrightarrow \Omega X$ and its adjoint $\Sigma(\Omega X)^{(t_\alpha)} \longrightarrow \Omega X$. This is true for every $\alpha \in \mathcal{I}$, so $\nabla \circ f^{m+1}(X)$ is null homotopic, implying that $\mathrm{wcocat}(X) \leq m$.

Finally, by Lemma 10.1 we obtain $\mathrm{wcocat}(BG) = m$ if and only if $\mathrm{nil}(G) = m$. $\square$

11. Retractile H-spaces

We now aim towards explicit calculations of the homotopy nilpotency class of quasi-p-regular exceptional Lie groups and nonmodular p-compact groups in Section 13. This requires introducing

a special family of finite H-spaces. Throughout this section we will assume that all spaces and maps have been localized (or completed) at a prime p, and homology is taken with mod-p coefficients.

Definition 11.1. Let B be an H-space. Suppose that there is a space A and a map $i\colon A \longrightarrow B$ satisfying the following:

(i) $H_*(B) \cong \Lambda(\widetilde{H}_*(A))$;

(ii) i_* induces the inclusion of the generating set;

(iii) Σi has a left homotopy inverse.

Then the triple (A, i, B) is a *retractile* H-space.

If (A, i, B) is a retractile H-space then many properties of B are determined by the restriction to A [GT1, Th1, Th2]. We will show this is also the case for $\mathrm{nil}(G)$ when G is a finite loop space and (A, i, G) is retractile. A large family of retractile H-spaces was constructed in different ways by Cooke, Harper and Zabrodsky [CHZ] and Cohen and Neisendorfer [CN]. They show that if p is a prime and A is a finite CW-complex consisting of ℓ odd dimensional cells, where $\ell < p-1$, then there exists a p-local H-space B and a map $i\colon A \longrightarrow B$ which makes (A, i, B) retractile. Further, in the boundary case when $\ell = p-1$, they show that if there happens to be a finite H-space B and a map $A \xrightarrow{i} B$ inducing an isomorphism $H_*(B) \cong \Lambda(\widetilde{H}_*(A))$, then Σi has a left homotopy inverse so (A, i, B) is retractile. Notice that products of retractile H-spaces are retractile. For if each of $\{(A_j, i_j, B_j)\}_{j=1}^m$ is retractile then so is $(\bigvee_{j=1}^m A_j, \bigvee_{i=1}^m i_j, \prod_{j=1}^m B_j)$.

Specific examples of retractile H-spaces are given by certain simply-connected simple compact Lie groups and p-compact groups. Fix a prime p. By [MNT], when localized at p every simply-connected, simple compact Lie group G decomposes as a product of $p-1$ spaces. If the rank is low with respect to p then by [Th2] the factors are all retractile, and so their product - the original group G - is also retractile. The precise ranks involved are as follows:

$$\begin{array}{ll} SU(n) & \text{if } n \leq (p-1)^2 + 1 \\ Sp(n) & \text{if } 2n \leq (p-1)^2 \\ Spin(2n+1) & \text{if } 2n \leq (p-1)^2 \\ Spin(2n) & \text{if } 2(n-1) \leq (p-1)^2 \\ G_2, F_4, E_6 & \text{if } p \geq 5 \\ E_7, E_8 & \text{if } p \geq 7. \end{array} \tag{12}$$

Recall that X is a p-compact group if $X \simeq \Omega BX$ where BX is a p-complete space and the mod-p homology of X is finite. There is a classification of irreducible p-compact groups [AGMV] when p is odd, which in conjunction with work in [BM, C], show that there are four familes: (i) simply-connected, simple compact Lie groups; (ii) an infinite family of spaces that decompose as a product of spheres, sphere bundles over spheres, and factors of $SU(n)$; (iii) 30 special nonmodular cases; and (iv) 4 exotic modular cases. The retractile cases in (i) are covered by (12). The retractile cases

in (ii) depend on $SU(n)$ being retractile, and from [D] it follows that every case in (iii) and (iv) is retractile. So to (12) we add a second list:

(13)
- (a) an infinite family of p-compact groups that decompose as a product of spheres, sphere bundles over spheres, and factors of $SU(n)$ for $n \leq (p-1)^2+1$;
- (b) the 30 special nonmodular p-compact groups (see [D, Table 3.2]);
- (c) the 4 exotic modular p-compact groups.

Since G is a loop space, it has a classifiying space BG and there is a homotopy equivalence $e\colon G \longrightarrow \Omega BG$. Let $\overline{ev}$ be the composite

$$\overline{ev}\colon \Sigma G \xrightarrow{\Sigma e} \Sigma\Omega BG \xrightarrow{ev} BG$$

where ev is the canonical evaluation map. Let j be the composite

$$j\colon \Sigma A \xrightarrow{\Sigma i} \Sigma G \xrightarrow{\overline{ev}} BG.$$

A key property is the following.

Lemma 11.2. *Let G be a group from list (12) or (13). Then there is a homotopy commutative diagram*

$$\begin{tikzcd} \Sigma G \arrow[r, "\overline{ev}"] \arrow[d, "r"] & BG \arrow[d, equal] \\ \Sigma A \arrow[r, "j"] & BG \end{tikzcd}$$

where r is a left homotopy inverse of Σi.

Proof. By hypothesis, the map $\Sigma A \xrightarrow{\Sigma i} \Sigma G$ has a right homotopy inverse, so $\Sigma G \simeq \Sigma A \vee C$ for some space C. In [GT1] the retractile properties were used to precisely describe the complementary factor C in the case when G is from the list (12), but the proof adapts immediately to list (13) as well. It is well known that there is a homotopy fibration $\Sigma G \wedge G \xrightarrow{\mu^*} \Sigma G \xrightarrow{\overline{ev}} BG$ where μ^* is the canonical Hopf construction. In [GT1] it is shown there is a homotopy equivalence

$$\Sigma A \vee C \xrightarrow{\Sigma i + s} \Sigma G \tag{14}$$

where C is a retract of $\Sigma G \wedge G$ and s factors through the Hopf construction μ^*. Consequently, s composes trivially with $\overline{ev}$, so from the decomposition in (14) we obtain a homotopy commutative diagram

$$\begin{tikzcd} \Sigma G \arrow[r, "\overline{ev}"] \arrow[d, "r"] & BG \arrow[d, equal] \\ \Sigma A \arrow[r, "j"] & BG \end{tikzcd}$$

where r is a left homotopy inverse of Σi. $\square$

For a space Y, let $E\colon Y \longrightarrow \Omega\Sigma Y$ be the suspension map, which is defined as the (right) adjoint of the identity map on ΣY. Notice that the evaluation map $\Sigma\Omega Y \xrightarrow{ev} Y$ is the (left) adjoint of the identity map on ΩY. So the composite $\Omega Y \xrightarrow{E} \Omega\Sigma\Omega Y \xrightarrow{\Omega ev} \Omega Y$ is homotopic to the identity map. This leads to the following corollary of Lemma 11.2.

Corollary 11.3. *The composite* $G \xrightarrow{E} \Omega\Sigma G \xrightarrow{\Omega r} \Omega\Sigma A \xrightarrow{\Omega j} \Omega BG$ *is homotopic to the homotopy equivalence* e*.*

Proof. By Lemma 11.2, $j \circ r \simeq \overline{ev} = ev \circ \Sigma e$. Thus $\Omega j \circ \Omega r \circ E \simeq \Omega ev \circ \Omega\Sigma e \circ E$. The naturality of E implies that the latter composite is homotopic to $\Omega ev \circ E \circ e$. But $\Omega ev \circ E$ is homotopic to the identity map on ΩBG, so we obtain $\Omega j \circ \Omega r \circ E \simeq e$, as asserted. $\square$

12. Cocategory and Nilpotence II

In this section a restricted version of Theorem 1.6 is proved.

Definition 12.1. Let G be an H-group. Suppose that Y is a pointed, path-connected space and there is a map $f\colon Y \longrightarrow G$. We say that G has *homotopy nilpotency class* m *with respect to* f if the composite $Y^{(m+1)} \xrightarrow{f^{(m+1)}} G^{(m+1)} \xrightarrow{c_m} G$ is null homotopic but $c_{m-1} \circ f^{(m)}$ is nontrivial. In this case we write $\mathrm{nil}(Y, f, G) = m$.

Definition 12.2. Let X and Y be pointed, path-connected spaces, and suppose that there is a map $g\colon Y \longrightarrow X$. We say that X has *weak cocategory* m *with respect to* g if the composite $F_{m+1}(Y) \xrightarrow{f_{m+1}(Y)} \bigvee_{i=1}^{m+1} Y \xrightarrow{\nabla} Y \xrightarrow{g} X$ is null homotopic but $g \circ \nabla \circ f_m(X)$ is nontrivial. In this case we write $\mathrm{wcocat}(Y, f, X) = m$.

Theorem 12.3. *Let G be a group from list (12) or (13). Then* $\mathrm{wcocat}(\Sigma A, j, BG) = m$ *if and only if* $\mathrm{nil}(A, i, G) = m$*.*

Proof. The naturality of the thin product and the naturality of the decomposition of $F^{m+1}(X)$ in Corollary 9.6 implies that there is a homotopy fibration diagram

$$\begin{tikzcd}
F^{m+1}(\Sigma A) \arrow[r, "f^{m+1}(\Sigma A)"] \arrow[d, "g^{m+1}(j)"] & \bigvee_{i=1}^{m+1} \Sigma A \arrow[r] \arrow[d, "\bigvee_{i=1}^{m+1} j"] & P^{m+1}(\Sigma A) \arrow[d, "P^{m+1}(j)"] \\
F_{m+1}(BG) \arrow[r, "f^{m+1}(BG)"] & \bigvee_{i=1}^{m+1} BG \arrow[r] & P^{m+1}(BG)
\end{tikzcd}$$

where $F^{m+1}(\Sigma A) \simeq \bigvee_{\alpha\in\mathcal{I}} \Sigma(\Omega\Sigma A)^{(t_\alpha)}$, $F^{m+1}(BG) \simeq \bigvee_{\alpha\in\mathcal{I}} \Sigma(\Omega BG)^{(t_\alpha)}$, and $g^{m+1}(j) \simeq \bigvee_{\alpha\in\mathcal{I}} \Sigma(\Omega j)^{(t_\alpha)}$.

Suppose that $\mathrm{wcocat}(\Sigma A, j, BG) = m$. Then the composite $F^{m+1}(\Sigma A) \xrightarrow{f^{m+1}(\Sigma A)} \bigvee_{i=1}^{m+1} \Sigma A \xrightarrow{\nabla} \Sigma A \xrightarrow{j} BG$ is null homotopic. Consider the Samelson product $A^{(m+1)} \xrightarrow{i^{(m+1)}} G^{(m+1)} \xrightarrow{c_m} G$. Its adjoint is the composite $\Sigma A^{(m+1)} \xrightarrow{\Sigma i^{(m+1)}} \Sigma G^{(m+1)} \xrightarrow{w_m} BG$, where w_m is the $(m+1)$-fold Whitehead product of $\overline{ev}$ with itself, which is adjoint to c_m.

We claim that there is a lift

$$
\begin{tikzcd}
 & & F^{m+1}(\Sigma A) \arrow[d, "f^{m+1}(\Sigma A)"] \\
 & & \bigvee_{i=1}^{m+1} \Sigma A \arrow[d, "j\circ\nabla"] \\
\Sigma A^{(m+1)} \arrow[r, "\Sigma i^{(m+1)}"] \arrow[rruu, dashed] & \Sigma G^{(m+1)} \arrow[r, "w_m"] & BG.
\end{tikzcd} \tag{15}
$$

The lift is constructed in two stages. First, we give a specific lift of $w_m \circ \Sigma i^{(m+1)}$ through $j \circ \nabla$. Let $s'_j\colon \Sigma A \longrightarrow \bigvee_{i=1}^{m+1} \Sigma A$ and $s_j\colon BG \longrightarrow \bigvee_{i=1}^{m+1} BG$ be the inclusions of the j^{th}-wedge summands, and let t'_j and t_j be the composites $t'_j\colon \Sigma\Omega\Sigma A \xrightarrow{ev} \Sigma A \xrightarrow{s'_j} \bigvee_{i=1}^{m+1} \Sigma A$ and $t_j\colon \Sigma G \xrightarrow{\overline{ev}} BG \xrightarrow{s_j} \bigvee_{i=1}^{m+1} BG$. Let $\overline{w}'_m = [t'_1, [t'_2, \ldots, [t'_m, t'_{m+1}] \ldots]$ and $\overline{w}_m = [t_1, [t_2, \ldots, [t_m, t_{m+1}] \ldots]$. Since $G \simeq \Omega BG$, for convenience we write $\Omega\Sigma A \xrightarrow{\Omega j} \Omega BG$ as $\Omega\Sigma A \xrightarrow{\Omega j} G$. Consider the diagram

$$
\begin{tikzcd}
\Sigma A^{(m+1)} \arrow[r, "\Sigma E^{(m+1)}"] \arrow[dr, "\Sigma i^{(m+1)}"'] & \Sigma\Omega\Sigma A^{(m+1)} \arrow[r, "\overline{w}'_m"] \arrow[d, "\Sigma(\Omega j)^{(m+1)}"] & \bigvee_{i=1}^{m+1} \Sigma A \arrow[r, "\nabla"] \arrow[d, "\bigvee_{i=1}^{m+1} j"] & \Sigma A \arrow[d, "j"] \\
 & \Sigma G^{(m+1)} \arrow[r, "\overline{w}_m"] & \bigvee_{i=1}^{m+1} BG \arrow[r, "\nabla"] & BG.
\end{tikzcd} \tag{16}
$$

The left triangle homotopy commutes since $i \simeq j \circ E$. The middle square homotopy commutes by the naturality of the Whitehead product. The right square homotopy commutes by the naturality of the fold map. Thus the entire diagram homotopy commutes. Observe that since $\Sigma A \xrightarrow{\Sigma E} \Sigma\Omega\Sigma A \xrightarrow{ev} \Sigma A$ is homotopic to the identity map, we have $t'_j \circ E \simeq s'_j$. Thus the composite $\overline{w}'_m \circ \Sigma E^{(m+1)}$ along the top row in (16) homotopic to $[s'_1, [s'_2, \ldots, [s'_m, s'_{m+1}] \ldots]$. On the other hand, since $w_m \simeq \nabla \circ \overline{w}_m$, the bottom direction around (16) is homotopic to $w_m \circ \Sigma i^{(m+1)}$. Thus the homotopy commutativity of (16) implies that $[s'_1, [s'_2, \ldots, [s'_m, s'_{m+1}] \ldots]$ lifts $w_m \circ \Sigma i^{(m+1)}$ through $j \circ \nabla$. Further, as $\overline{w}'_m$ is an $(m+1)$-fold Whitehead product involving all $m+1$ spaces in the wedge $\bigvee_{i=1}^{m+1} \Sigma A$, Theorem 9.1 implies that $\overline{w}'_m$ lifts through $f^{m+1}(\Sigma A)$ to $F^{m+1}(\Sigma A)$.

By hypothesis, $\mathrm{wcocat}(\Sigma A, j, BG) = m$, so the composite $F^{m+1}(\Sigma A) \xrightarrow{f^{m+1}(\Sigma A)} \bigvee_{i=1}^{m+1} \Sigma A \xrightarrow{\nabla} \Sigma A \xrightarrow{j} BG$ is null homotopic. Therefore the lift of $w_m \circ \Sigma i^{(m+1)}$ through $\nabla \circ j \circ f^{m+1}(\Sigma A)$ in (15) implies that $w_m \circ \Sigma i^{(m+1)}$ is null homotopic. Taking adjoints, $w_m \circ i^{(m+1)}$ is null homotopic, and so $\mathrm{nil}(A, i, G) \leq m$.

Conversely, suppose that $\mathrm{nil}(A, i, G) \leq m$. Then the composite $A^{(m+1)} \xrightarrow{i^{(m+1)}} G^{(m+1)} \xrightarrow{c_m} G$ is null homotopic. Taking adjoints, the composite $\Sigma A^{(m+1)} \xrightarrow{\Sigma i^{(m+1)}} \Sigma G^{(m+1)} \xrightarrow{w_m} BG$ is null homotopic. Consider the composite $F^{m+1}(\Sigma A) \xrightarrow{f^{m+1}(\Sigma A)} \bigvee_{i=1}^{m+1} \Sigma A \xrightarrow{\nabla} \Sigma A \xrightarrow{j} BG$. By Corollary 9.6 we have $F^{m+1}(\Sigma A) \simeq \bigvee_{\gamma \in \mathcal{J}_{m+1}} \Sigma A^{(t_\gamma)}$ and $f^{m+1}(\Sigma A)$ restricted to $\Sigma A^{(t_\gamma)}$ is an iterated Whitehead product w_{t_γ} of length $\geq m+1$ formed from the inclusions $\Sigma A \hookrightarrow \bigvee_{i=1}^{m+1} \Sigma A$ of the wedge summands. Thus each w_{t_γ} factors through $w_m \circ \Sigma i^{(m+1)}$, which by hypothesis is null homotopic. Therefore w_{t_γ}

is null homotopic. As this is true for every wedge summand of $F^{m+1}(\Sigma A)$, the map $f^{m+1}(\Sigma A)$ is null homotopic. Thus $j \circ \nabla \circ f^{m+1}(\Sigma A)$ is null homotopic, implying that $\mathrm{wcocat}(\Sigma A, j, BG) \leq m$.

Hence, by Lemma 10.1 we obtain $\mathrm{wcocat}(\Sigma A, j, BG) = m$ if and only if $\mathrm{nil}(A, i, G) = m$. □

The next theorem links the equivalences in Theorems 1.6 and 12.3.

Theorem 12.4. *Let G be a group from list (12) or (13). Then* $\mathrm{wcocat}(BG) = m$ *if and only if* $\mathrm{wcocat}(\Sigma A, j, BG) = m$.

Proof. In general, by Corollary 9.6, for a space X there is a homotopy fibration

$$F^{m+1}(X) \xrightarrow{f^{m+1}(X)} \bigvee_{i=1}^{m+1} X \longrightarrow P^m(X) \tag{17}$$

where the right map is the inclusion of the wedge into the thin product; $F^{m+1}(X) \simeq \bigvee_{\alpha\in\mathcal{I}} \Sigma(\Omega X)^{(t_\alpha)}$; and $f^{m+1}(X)$ is a wedge sum of iterated Whitehead products of maps of the form $\Sigma\Omega X \xrightarrow{ev} X \xrightarrow{s_j} \bigvee_{i=1}^{m+1} X$. Moreover, all of this is natural for maps of spaces $X \longrightarrow Y$. Recall that (A, i, G) being retractile means that there is a map $\Sigma G \xrightarrow{r} \Sigma A$ which is a left homotopy inverse of Σi. Starting from the composite $\Sigma G \xrightarrow{r} \Sigma A \xrightarrow{j} BG$, consider the diagram

$$\begin{tikzcd}
\bigvee_{\alpha\in\mathcal{I}} \Sigma G^{(t_\alpha)} \arrow[rr, "\simeq"] \arrow[d, "\bigvee_{\alpha\in\mathcal{I}} \Sigma E^{(t_\alpha)}"] & & \bigvee_{\alpha\in\mathcal{I}} \Sigma(\Omega BG)^{(t_\alpha)} \arrow[d, equal] \\
\bigvee_{\alpha\in\mathcal{I}} \Sigma(\Omega\Sigma G)^{(t_\alpha)} \arrow[r, "\bigvee_{\alpha\in\mathcal{I}} \Sigma(\Omega r)^{(t_\alpha)}"] \arrow[d, "\simeq"] & \bigvee_{\alpha\in\mathcal{I}} \Sigma(\Omega\Sigma A)^{(t_\alpha)} \arrow[r, "\bigvee_{\alpha\in\mathcal{I}} \Sigma(\Omega j)^{(t_\alpha)}"] \arrow[d, "\simeq"] & \bigvee_{\alpha\in\mathcal{I}} \Sigma(\Omega BG)^{(t_\alpha)} \arrow[d, "\simeq"] \\
F^{m+1}(\Sigma G) \arrow[r] \arrow[d, "f^{m+1}(\Sigma G)"] & F^{m+1}(\Sigma A) \arrow[r] \arrow[d, "f^{m+1}(\Sigma A)"] & F^{m+1}(BG) \arrow[d, "f^{m+1}(BG)"] \\
\bigvee_{i=1}^{m+1} \Sigma G \arrow[r, "\bigvee_{i=1}^{m+1} r"] \arrow[d, "\nabla"] & \bigvee_{i=1}^{m+1} \Sigma A \arrow[r, "\bigvee_{i=1}^{m+1} j"] \arrow[d, "\nabla"] & \bigvee_{i=1}^{m+1} BG \arrow[d, "\nabla"] \\
\Sigma G \arrow[r, "r"] & \Sigma A \arrow[r, "j"] & BG.
\end{tikzcd} \tag{18}$$

The bottom squares homotopy commute by the naturality of the fold map, the middle and top squares homotopy commute by the naturality of (17) and the naturality of the decomposition of $F^{m+1}(X)$. By Corollary 11.3, $\Omega j \circ \Omega r \circ E$ is a homotopy equivalence, so the top rectangle also homotopy commutes.

Suppose that $\mathrm{wcocat}(BG) = m$. Then the composite $F^{m+1}(BG) \xrightarrow{f^{m+1}(BG)} \bigvee_{i=1}^{m+1} BG \xrightarrow{\nabla} BG$ is null homotopic. The homotopy commutativity of (18) therefore implies that the composite $F^{m+1}(\Sigma A) \xrightarrow{f^{m+1}(\Sigma A)} \bigvee_{i=1}^{m+1} \Sigma A \xrightarrow{\nabla} \Sigma A \xrightarrow{j} BG$ is null homotopic. Thus $\mathrm{wcocat}(\Sigma A, j, BG) \leq m$.

Suppose that $\mathrm{wcocat}(\Sigma A, j, BG) = m$. Then the composite $F^{m+1}(\Sigma A) \xrightarrow{f^{m+1}(\Sigma A)} \bigvee_{i=1}^{m+1} \Sigma A \xrightarrow{\nabla} \Sigma A \xrightarrow{j} BG$ is null homotopic. The homotopy commutativity of (18) therefore implies that the upper direction around the diagram is null homotopic. But as the top row is a homotopy equivalence, this

implies that the composite $F^{m+1}(BG) \xrightarrow{f^{m+1}(BG)} \bigvee_{i=1}^{m+1} BG \xrightarrow{\nabla} BG$ is null homotopic. That is, $\mathrm{wcocat}(BG) \leq m$.

Hence, by Lemma 10.1 we obtain $\mathrm{wcocat}(BG) = m$ if and only if $\mathrm{wcocat}(\Sigma A, j, BG) = m$. $\square$

Combining the equivalences in Theorems 1.6, 12.3 and 12.4 we obtain the following.

Theorem 12.5. *Let G be a group from (12) or (13). Then* $\mathrm{nil}(G) = m$ *if and only if* $\mathrm{nil}(A, i, G) = m$. $\square$

This reduction from checking whether $G^{(m+1)} \xrightarrow{c_m} G$ is null homotopic to checking whether $A^{(m+1)} \xrightarrow{i^{(m+1)}} G^{(m+1)} \xrightarrow{c_m} G$ is null homotopic gives a very practical tool for determining the homotopy nilpotency class of certain finite loop spaces. This will be used in the next section to give explicit calculations.

13. Examples

Fix an odd prime p. Let G be a p-localized simply-connected simple compact Lie group, or in the p-complete setting, let $G^{\wedge}_p$ be a p-compact group. In the p-local case, G is rationally homotopy equivalent to a product of odd dimensional spheres $\prod_{i=1}^{\ell} S^{2n_i-1}$, and in the p-complete case, $G^{\wedge}_p$ is rationally homotopy equivalent to the rationalization of a product of odd dimensional p-completed spheres $\prod_{i=1}^{\ell} (S^{2n_i-1})^{\wedge}_p$. Reordering the indices if need be, we may assume that $n_1 \leq \cdots \leq n_\ell$. The *type* of G is $\{n_1, \ldots, n_\ell\}$. From now on we drop the usual $(\ \ )^{\wedge}_p$ notation for p-completion for convenience, the context making it clear when it should be used. In the p-local (p-complete) case we say that G is *p-regular* if there is a p-local (or p-complete) homotopy equivalence $G \simeq \prod_{i=1}^{\ell} S^{2n_i-1}$. It is *quasi-$p$-regular* if there is a p-local (or p-complete) homotopy equivalence $G \simeq \prod_{i=1}^{\ell} B_i$, where each B_i is either a sphere or a sphere bundle over a sphere.

Kaji and Kishimoto [KK] have determined the homotopy nilpotency classes of all p-regular p-compact groups. Kishimoto [K] determined the homotopy nilpotency class of all quasi-p-regular cases of $SU(n)$. In both cases the calculations were intense and involved detailed information about nontrivial Samelson products on the group in question. In contrast, very little is known about nontrivial Samelson products in exceptional Lie groups, so one would ordinarily not hope to be able to determine their homotopy nilpotency classes. Nevertheless, in Theorem 13.1 we will calculate the homotopy nilpotency class in most quasi-p-regular cases.

The sphere bundles over spheres that appear as factors have a particular form, which we describe first. Let $\pi^S_m(S^n)$ be the m^{th}-stable homotopy group of S^n. The least nonvanishing p-torsion homotopy group of S^3 is $\pi_{2p}(S^3) \cong \mathbb{Z}/p\mathbb{Z}$. Let $\alpha\colon S^{2p} \longrightarrow S^3$ represent a generator. This map is stable and it represents the generator in $\pi^S_{n+2p-3}(S^n) \cong \mathbb{Z}/p\mathbb{Z}$. For $n \geq 1$, define the space

$B(2n+1, 2n+2p-1)$ by the homotopy pullback

$$\begin{tikzcd} S^{2n+1} \arrow[r] \arrow[d, equal] & B(2n+1,2n+2p-1) \arrow[r] \arrow[d] & S^{2n+2p-1} \arrow[d, "\alpha"] \\ S^{2n+1} \arrow[r] & S^{4n+3} \arrow[r, "w"] & S^{2n+2} \end{tikzcd}$$

where w is the Whitehead product of the identity map with itself.

Restrict to $p \geq 7$ and consider the exceptional simple compact Lie groups which are quasi-p-regular but not p-regular. By [MNT] a complete list is as follows:

$$\begin{array}{lll} F_4 & p=7 & B(3,15)\times B(11,23) \\ & p=11 & B(3,23)\times S^{11}\times S^{15} \\ E_6 & p=7 & F_4\times S^9\times S^{17} \\ & p=11 & F_4\times S^9\times S^{17} \\ E_7 & p=11 & B(3,23)\times B(15,35)\times S^{11}\times S^{19}\times S^{27} \\ & p=13 & B(3,27)\times B(11,35)\times S^{15}\times S^{19}\times S^{23} \\ & p=17 & B(3,35)\times S^{11}\times S^{15}\times S^{19}\times S^{23}\times S^{27} \\ E_8 & p=11 & B(3,23)\times B(15,35)\times B(27,47)\times B(39,59) \\ & p=13 & B(3,27)\times B(15,39)\times B(23,47)\times B(35,59) \\ & p=17 & B(3,35)\times B(15,47)\times B(27,59)\times S^{23}\times S^{39} \\ & p=19 & B(3,39)\times B(23,59)\times S^{15}\times S^{27}\times S^{35}\times S^{47} \\ & p=23 & B(3,47)\times B(15,59)\times S^{23}\times S^{27}\times S^{35}\times S^{39} \\ & p=29 & B(3,59)\times S^{15}\times S^{23}\times S^{27}\times S^{35}\times S^{39}\times S^{47}. \end{array} \tag{19}$$

McGibbon [M] showed that none of the Lie groups listed in (19) is homotopy commutative. Thus, in every case, $\mathrm{nil}(G) \geq 2$.

Complete at $p \geq 7$ and consider the nonmodular p-compact groups in cases 4 through 34 of the Shepard-Todd numbering which are quasi-p-regular but not p-regular. We exclude case 28 which

is F_4. As in [D], a complete list is given by:

(20)

Case	*Prime*	*Space*
5	7	$B(11,23)$
9	17	$B(15,47)$
10	13	$B(23,47)$
14	19	$B(11,47)$
16	11	$B(39,59)$
17	41	$B(39,119)$
18	31	$B(59,119)$
20	19	$B(23,59)$
24	11	$B(7,27) \times S^{11}$
25	7	$B(11,23) \times S^{17}$
26	13	$B(11,35) \times S^{23}$
27	19	$B(23,59) \times S^{13}$
29	13	$B(15,39) \times S^{7} \times S^{23}$
29	17	$B(7,39) \times S^{15} \times S^{23}$
30	11	$B(3,23) \times B(39,59)$
30	19	$B(3,39) \times S^{27} \times S^{59}$
30	29	$B(3,59) \times S^{23} \times S^{39}$
31	13	$B(15,39) \times B(23,47)$
31	17	$B(15,47) \times S^{23} \times S^{39}$
32	13	$B(23,47) \times B(35,59)$
32	19	$B(23,59) \times S^{35} \times S^{47}$
33	13	$B(11,35) \times S^{7} \times S^{19} \times S^{23}$
34	31	$B(23,83) \times S^{11} \times S^{35} \times S^{47} \times S^{59}$
34	37	$B(11,83) \times S^{23} \times S^{35} \times S^{47} \times S^{59}$

Saumell [Sa] showed that the only groups in (20) which are homotopy commutative are: $B(15,47)$ at $p=17$, $B(11,47)$ at $p=19$, $B(39,119)$ at $p=41$, $B(23,59)$ at $p=19$, and $B(7,27)\times S^{11}$ at $p=11$. Otherwise, the groups are not homotopy commutative, in which case $\mathrm{nil}(G)\geq 2$.

Theorem 13.1. *Let $p\geq 7$ and let G be a quasi-p-regular p-compact group from (19) or (20). If G is not homotopy commutative then* $\mathrm{nil}(G)=2$.

We begin with some properties of the space $B(2n+1,2n+2p-1)$, which has been well studied. It is a three-cell complex whose mod-p homology is $H_*(B(2n+1,2n+2p-1))\cong\Lambda(x_{2n+1},x_{2n+2p-1})$, where $|x_t|=t$. Let $A(2n+1,2n+2p-1)$ be the $(2n+2p-1)$-skeleton of B. Then

$$H_*(B(2n+1,2n+2p-1))\cong\Lambda(\widetilde{H}_*(A(2n+1,2np+1)))$$

and the skeletal inclusion $i\colon A(2n+1,2n+2p-1) \longrightarrow B(2n+1,2n+2p-1)$ induces the inclusion of the generating set in homology. By [CHZ, CN], Σi has a right homotopy inverse. Thus we obtain the following.

Lemma 13.2. *For $n \geq 1$, $(A(2n+1,2n+2p-1), i, B(2n+1,2n+2p-1))$ is a retractile triple.* $\square$

Let G be a quasi-p-regular p-compact group such that $G \simeq \prod_{i=1}^{\ell} B_i$, where each B_i is a sphere or space of the form $B(2n_i+1,2n_i+2p-1)$. If $B_i = S^{2n_i-1}$ let $A_i = S^{2n_i-1}$ and let $I_i\colon A_i \longrightarrow B_i$ be the identity map. If $B_i = B(2n_i+1,2n_i+2p-1)$ let $A_i = A(2n_i+1,2n_i+2p-1)$. Let $A = \bigvee_{i=1}^{\ell} A_i$ and let $I\colon A \longrightarrow B$ be the wedge sum of the maps I_i for $1 \leq i \leq \ell$. The retractile property in Lemma 13.2 implies that (A, I, G) is also retractile. Referring to G itself as being retractile, this is stated as follows.

Lemma 13.3. *Let G be a quasi-p-regular p-compact group which is a product of spheres and spaces of the form $B(2n+1,2n+2p-1)$. Then G is retractile.* $\square$

Some elementary information about the homotopy groups of G is also required. This requires a few preliminary lemmas.

Lemma 13.4. *Let X be a finite CW-complex with cells in dimensions $\{m_1,\ldots,m_s\}$. Let Y be a space with the property that $\pi_{m_i}(Y) = 0$ for $1 \leq i \leq s$. Then any map $f\colon X \longrightarrow Y$ is null homotopic.*

Proof. For a positive integer t, let X_t be the t-skeleton of X. Since X has cells in dimensions $\{m_1,\ldots,m_s\}$, for $1 \leq i \leq s$ there are cofibration sequences

$$\bigvee S^{m_i-1} \xrightarrow{f_i} X_{m_{i-1}} \longrightarrow X_{m_i} \xrightarrow{q_i} \bigvee S^{m_i}$$

for attaching maps f_i and pinch maps q_i, where X_{m_0} is the basepoint.

Clearly the restriction of $X \xrightarrow{f} Y$ to X_{m_0} is null homotopic. Suppose inductively that the restriction of f to $X_{m_{i-1}}$ is null homotopic. Then from the cofibration $X_{m_{i-1}} \longrightarrow X_{m_i} \xrightarrow{q_i} \bigvee S^{m_i}$ we see that the restriction of f to X_{m_i} factors through q_i to a map $e_i\colon \bigvee S^{m_i} \longrightarrow Y$. But as $\pi_{m_i}(Y) = 0$, the map e_i is null homotopic. Therefore the restriction of f to X_{m_i} is null homotopic. By induction, the restriction of f to $X_{m_s} = X$ - that is, f itself - is null homotopic. $\square$

A space B is spherically resolved by spheres $S^{n_1},\ldots,S^{n_t}$ if for $1 \leq j < t$ there are homotopy fibrations

$$S^{n_j} \longrightarrow B_j \longrightarrow B_{j+1}$$

where $B_1 = B$ and $B_t = S^{n_t}$. For example, $SU(n)$ is spherically resolved by $S^3, S^5, \ldots, S^{2n-1}$. From Lemma 13.4 we obtain the following refinement in the case of spherically resolved spaces.

Lemma 13.5. *Let X be a finite CW-complex with cells in dimensions $\{m_1,\ldots,m_s\}$. Let B be spherically resolved by spheres $S^{n_1},\ldots,S^{n_t}$. If $\pi_{m_i}(S^{n_j}) = 0$ for all $1 \leq i \leq s$ and $1 \leq j \leq t$, then any map $f\colon X \longrightarrow B$ is null homotopic.*

Proof. Proceed by downward induction on t. When $t = n$, as $B_t = S^{n_t}$, the hypotheses immediately imply that $\pi_{m_i}(B_t) = 0$ for $1 \leq i \leq s$. If $j < t$, suppose inductively that $\pi_{m_i}(B_{j+1}) = 0$ for $1 \leq i \leq s$. Then as $\pi_{m_i}(S^{n_j}) = 0$ for $1 \leq i \leq s$, the long exact sequence of homotopy groups induced by the homotopy fibration $S^{n_j} \longrightarrow B_j \longrightarrow B_{j+1}$ implies that $\pi_{m_i}(B_j) = 0$ for $1 \leq i \leq s$. As $B = B_1$, by induction we have $\pi_{m_i}(B) = 0$ for $1 \leq i \leq s$. Lemma 13.4 then implies that any map $X \xrightarrow{f} B$ is null homotopic. $\square$

A special case is $B(3, 2p+1)$. This can be regarded as the homotopy fibre of the map $\overline{\alpha}\colon S^{2p+1} \longrightarrow BS^3$ which is adjoint to $S^{2p} \xrightarrow{\alpha} S^3$. Since α has order p, so does its adjoint, and therefore $\overline{\alpha} \circ p$ is null homotopic, implying that the degree p map on S^{2p+1} lifts to the homotopy fibre of $\overline{\alpha}$. That is, there is a map $c\colon S^{2p+1} \longrightarrow B(3, 2p + 1)$ with the property that the composite $S^{2p+1} \xrightarrow{c} B(3, 2p + 1) \longrightarrow S^{2p+1}$ is of degree p. For a space X, let $X\langle 3\rangle$ be the three-connected cover of X. Notice that the map c lifts to a map $\bar{c}\colon S^{2p+1} \longrightarrow B(3, 2p + 1)\langle 3\rangle$. The following property was proved by Toda [To1].

Lemma 13.6. *The map $S^{2p+1} \xrightarrow{\bar{c}} B(3, 2p + 1)\langle 3\rangle$ is $(2p^2 - 2)$-connected. In particular, $\bar{c}$ induces an isomorphism on the homotopy groups π_m for $m \leq 2p^2 - 2$.* $\square$

Proposition 13.7. *Let p be an odd prime. Let G be a quasi-p-regular p-compact group which is a product of spheres and spaces of the form $B(2n + 1, 2n + 2p - 1)$. Suppose that either:*

(i) *G has type $\{n_1, \ldots, n_\ell\}$ where $3n_\ell < \min\{n_1 p, n_1 + p^2 - p\}$; or*

(ii) *$G \simeq B(3, 2p + 1) \times Y$ where Y is as in part (i) and its type also satisfies $2 < n_1 \leq p$ and $p < n_\ell$.*

Then $\mathrm{nil}(G) \leq 2$.

Proof. Part (i): By Lemma 13.3, the hypotheses on G imply that there is a retractile triple (A, i, G). By Theorem 12.5, to show that $\mathrm{nil}(G) \leq 2$ it suffices to show that the composite $A \wedge A \wedge A \xrightarrow{i\wedge i\wedge i} G \wedge G \wedge G \xrightarrow{c_2} G$ is null homotopic. We will show more, that any map $f\colon A \wedge A \wedge A \longrightarrow G$ is null homotopic.

Since G is quasi-p-regular of type $\{n_1, \ldots, n_\ell\}$, it is spherically resolved by the odd dimensional spheres $S^{2n_1-1}, \ldots, S^{2n_\ell-1}$. The type of G also implies that A has a CW-structure with ℓ cells, in dimensions $\{2n_1 - 1, \ldots, 2n_\ell - 1\}$. So $A \wedge A \wedge A$ has a CW-structure consisting of cells in dimensions $\{m_1, \ldots, m_s\}$ where each m_i is odd. Therefore, by Lemma 13.5, if $\pi_m(S^{2n_i-1}) = 0$ for each $m \in \{m_1, \ldots, m_s\}$ and each $1 \leq i \leq \ell$, then f is null homotopic.

The dimension of $A \wedge A \wedge A$ is $6n_\ell - 3$, so we consider $\pi_m(S^{2n_i-1})$ for m an odd number with $m \leq 6n_\ell - 3$. Observe that the stable range for S^{2n_i-1} is $2n_i p - 4$. That is, $\pi_m(S^{2n_i-1}) \cong \pi^S_m(S^{2n_i-1})$ if $m < 2n_i p - 3$. Since $n_1 \leq \cdots \leq n_\ell$, S^{2n_1-1} has the stable range of least dimension. So if $6n_\ell - 3 < 2n_1 p - 3$, then $\pi_m(S^{2n_i-1})$ is stable for every $m \in \{m_1, \ldots, m_s\}$ and every $1 \leq i \leq \ell$.

This inequality is equivalent to $3n_\ell < n_1 p$, which is one of the hypotheses. Therefore, we need only consider the stable homotopy groups $\pi_m^S(S^{2n_i-1})$.

Since m is odd, the stable homotopy group $\pi_m^S(S^{2n_i-1})$ is in an even stem. By [To2], the even stable stem of least dimension is $2p(p-1)-2$ (the stem of the stable generator β_1). Thus $\pi_m^S(S^{2n_i-1}) = 0$ for odd m whenever $m < (2n_i - 1) + 2p(p-1) - 2$. As $n_1 \leq \cdots \leq n_\ell$ and $m \leq 6n_\ell - 3$, we have $\pi_m^S(S^{2n_i-1}) = 0$ whenever $6n_\ell - 3 < (2n_1 - 1) + 2p(p-1) - 2$. This inequality is equivalent to $3n_\ell < n_1 + p^2 - p$, which is one of the hypotheses.

Hence $\pi_m(S^{2n_i-1}) = 0$ for each $m \in \{m_1, \ldots, m_s\}$ and for each $1 \leq i \leq n$, and the proof of part (i) is complete.

Part (ii): Since $G \simeq B(3, 2p+1) \times Y$ with Y as in part (i), by Lemma 13.3 there is a retractile triple (A, i, G) where $A = A_1 \vee A_2$ for retractile triples $(A_1, i_1, B(3, 2p+1))$ and (A_2, i_2, Y). Notice that A_1 is the $(2p+1)$-skeleton of $B(3, 2p+1)$.

Consider the composite $\theta\colon A \wedge A \wedge A \xrightarrow{i \wedge i \wedge i} G \wedge G \wedge G \xrightarrow{c_2} G$. Observe that $A \wedge A \wedge A$ is 8-connected, so θ factors through the three-connected cover $G\langle 3\rangle$. Since $G \simeq B(3, 2p+1) \times Y$ and $n_1 > 2$ implies that Y is at least 4-connected, we have $G\langle 3\rangle \simeq B(3, 2p+1)\langle 3\rangle \times Y$. By Lemma 13.6, $\pi_k(B(3, 2p+1)\langle 3\rangle) \cong \pi_k(S^{2p+1})$ for $k < 2p^2 - 1$. Thus, in this dimensional range, it is as if G is spherically resolved by $S^{2n_1-1}, \ldots, S^{2n_\ell-1}$ and S^{2p+1}. The hypothesis that $p < n_\ell$ implies that the dimension of $A \wedge A \wedge A$ is that of $A_2 \wedge A_2 \wedge A_2$, which is $6n_\ell - 3$. The hypothesis that $n_1 \leq p$ implies that the arguments for $\pi_m(S^{2n_i-1}) = 0$ in part (1) also imply that $\pi_m(S^{2p+1}) = 0$. Thus if $6n_\ell - 3 < 2p^2 - 1$, then the argument in part (i) goes through without change to show that θ is null homotopic and $\mathrm{nil}(G) \leq 2$.

It remains to show that $6n_\ell - 3 < 2p^2 - 1$. By hypothesis, $3n_\ell < \min\{n_1 p, n_1 + p^2 - p\}$ and $n_1 \leq p$. If $\min\{n_1 p, n_1 + p^2 - p\} = n_1 p$ then $3n_\ell < n_1 p$ and $n_1 \leq p$ implies that $6n_\ell - 3 < 2p^2 - 3 < 2p^2 - 1$. If $\min\{n_1 p, n_1 + p^2 - p\} = n_1 + p^2 - p$ then $3n_\ell < n_1 + p^2 - p$ and $n_1 \leq p$ implies that $3n_\ell < p^2$ and so $6n_\ell - 3 < 2p^2 - 3 < 2p^2 - 1$. In either case, $6n_\ell - 3 < 2p^2 - 1$, as required. $\square$

Proof of Theorem 13.1. It has already been mentioned that every group G in (19) and every non-homotopy commutative group G in (20) has $\mathrm{nil}(G) \geq 2$. In each such case, except E_8 at $p = 11$, the hypotheses of Proposition 13.7 hold, so $\mathrm{nil}(G) \leq 2$. Hence $\mathrm{nil}(G) = 2$.

The one outstanding case is E_8 at $p = 11$. As in part (ii) of Proposition 13.7, write $E_8 \simeq B(3, 23) \times Y$ where Y has type $\{8, 14, 18, 20, 24, 30\}$. Now $n_1 = 8$ and $n_\ell = 30$, but $3n_\ell$ is not less than $\min\{n_1 p, n_1 + p^2 - p\} = \min\{88, 118\}$. The one obstruction that appears in the argument proving Proposition 13.7 corresponds to $6n_\ell - 3 = 177$ (the dimension of $A \wedge A \wedge A$) being less than $2n_1 p - 3 = 173$ (the stable range of S^{15}). But from the homotopy fibration $W_8 \longrightarrow S^{15} \xrightarrow{E^2} \Omega^2 S^{17}$ induced by the double suspension E^2, since W_8 is homotopy equivalent to the Moore space $P^{175}(11)$

in dimensions ≤ 180 and $\pi_{177}(P^{175}(11)) = 0$, we see that $\pi_{177}(S^{15})$ is stable, and so the argument in Proposition 13.7 goes through. Therefore $\mathrm{nil}(E_8) = 2$ in this case as well. $\square$

Remark 13.8. There are other quasi-p-regular exceptional Lie groups: G_2, F_4 and E_6 at $p = 3$ and $p = 5$. The cases F_4 and E_6 at 3 have torsion in mod-3 homology; therefore a characterization of homotopy nilpotent Lie groups by Rao [R] implies that, localized at 3, we have $\mathrm{nil}(F_4) = \infty$ and $\mathrm{nil}(E_6) = \infty$. McGibbon [M] showed that G_2 at $p = 5$ is homotopy commutative; therefore localized at 5 we have $\mathrm{nil}(G_2) = 1$. The remaining cases all have torsion free homology and are not homotopy commutative. Moreover, they do not satisfy the hypotheses of Proposition 13.7 and fail to do so in a way that does not allow for a sidestepping argument as for E_8 at $p = 11$ in the proof of Theorem 13.1. So the precise value of $\mathrm{nil}(G)$ in these cases remains undetermined.

References

[AGMV] K.K.S. Andersen, J. Grodal, J.M. Møller and A. Viruel, The classification of p-compact groups for p odd, *Ann. Math.* **167** (2008), 95-210.

[A] D. Anick, Differential algebras in topology, Research Notes in Math. 3, AK Peters, 1993.

[BBCG] A. Bahri, M. Bendersky, F.R. Cohen, and S. Gitler, The polyhedral product functor: a method of decomposition for moment-angle complexes, arrangements and related spaces, *Adv. Math.* **225** (2010), 1634-1668.

[BG] P. Beben and J. Grbić, Configuration spaces and polyhedral products, arXiv:1409.4462.

[BM] C. Broto and J.M. Møller, Chevalley p-local finite groups, *Algebr. Geom. Topol.* **7** (2007), 1809-1919.

[C] N. Castellana, Quillen Grassmannians as non-modular homotopy fixed points, *Math. Z.* **248** (2004), 477-493.

[CS] B. Chorny and J. Scherer, Goodwillie calculus and Whitehead products, arXiv:1109.2691.

[CMN] F.R. Cohen, J.C. Moore and J.A. Neisendorfer, Torsion in homotopy groups, *Ann. of Math.* **109** (1979), 121-168.

[CN] F.R. Cohen and J.A. Neisendorfer, A construction of p-local H-spaces, pp. 351-359. Lecture Notes in Math. Vol. 1051, Springer, Berlin, 1984.

[CHZ] G. Cooke, J. Harper, and A. Zabrodsky, Torsion free mod p H-spaces of low rank, *Topology* **18** (1979), 349-359.

[D] D.M. Davis, Homotopy type and v_1-periodic homotopy groups of p-compact groups, *Topology Appl.* **156** (2008), 300-321.

[DS] G. Denham and A.I. Suciu, Moment-angle complexes, monomial ideals, and Massey products, *Pure Appl. Math. Q.* **3** (2007), 25-60.

[G1] T. Ganea, Lusternik-Schnirelmann category and cocategory, *Proc. London. Math. Soc.* **10** (1960), 623-639.

[G2] T. Ganea, A generalization of the homology and homotopy suspension, *Comment. Math. Helv.* **39** (1965), 295-322.

[GT1] J. Grbić and S. Theriault, Odd primary self-maps of low rank Lie groups, *Canad. J. Math* **62** (2010), 284-304.

[GT2] J. Grbić and S. Theriault, The homotopy type of the complement of a coordinate subspace arrangement, *Topology* **46** (2007), 357-396.

[GT3] J. Grbić and S. Theriault, The homotopy type of the polyhedral product for shifted complexes, *Adv. Math.* **245** (2013), 690-715.

[Hop] M.J. Hopkins, Formulations of cocategory and the iterated suspension, *Algebraic homotopy and local algebra (Luminy, 1982)*, 212-226, Astérisque **113-114** Soc. Math. France, Paris, 1984.

[Hov] M. Hovey, Lusternik-Schnirelmann cocategory, *Illinois J. Math* **37** (1993), 224-239.

[IK1] K. Iriye and D. Kishimoto, Decompositions of polyhedral products, *Adv. Math.* **245** (2013), 716-736.

[IK2] K. Iriye and D. Kishimoto, The fat wedge filtration and a homotopy decomposition of a polyhedral product, arXiv:1412.4866

[J] I.M. James, Reduced product spaces, *Ann. of Math.* **62** (1955), 170-197.

[KK] S. Kaji and D. Kishimoto, Homotopy nilpotency in p-regular loop spaces, *Math Z.* **264** (2010), 209-224.

[K] D. Kishimoto, Homotopy nilpotency in localized $SU(n)$, *Homology, Homotopy Appl.* **11** (2009), 61-79.

[KM] B.B. el Krafi and M.I. Mamouni, On the rational cocategory and nilpotency of self homotopy equivalences and associated classifying spaces, preprint.

[M] C.A. McGibbon, Homotopy commutativity in localized groups, *Amer. J. Math* **106** (1984), 665-687.

[MNT] M. Mimura, G. Nishida, and H. Toda, Mod p Decomposition of Compact Lie Groups, *Publ. Res. Inst. Math. Sci.* **13** (1977), 627-680.

[MV] A. Murillo and A. Viruel, Lusternik-Schnirelmann cocategory: a Whitehead dual approach, *Cohomological methods in homotopy theory (Bellaterra, 1998)*, 323-347, Progress in Math. **196** Birkhäuser, Basel, 2001.

[N] J.A. Neisendorfer, *Algebraic methods in unstable homotopy theory*, New Mathematical Monographs vol. 12, Cambridge University Press, Cambridge, 2009.

[P] G. Porter, The homotopy groups of wedges of suspensions, *Amer. J. Math.* **88** (1966), 655-663.

[R] V.K. Rao, Homotopy nilpotent Lie groups have no torsion in homology, *Manuscripta Math.* **92** (1997), 455-462.

[Sa] L. Saumell, Homotopy commutativity of finite loop spaces, *Math. Z.* **207** (1991), 319-334.

[Sb] S.M. Sbaï, Modèle de Quillen des espaces de Ganea et cocatégorie rationelle, *C.R. Acad. Sci. Paris Sér. I. Math.* **332** (2001), 821-824.

[Se] P. Selick, *Introduction to homotopy theory*, Fields Institute Monographs **9**, American Mathematical Society, Providence, RI, 1997.

[Th1] S.D. Theriault, The H-structure of low rank torsion free H-spaces, *Quart. J. Math. Oxford* **56** (2005), 403-415.

[Th2] S.D. Theriault, The odd primary H-structure of low rank Lie groups and its application to exponents, *Trans. Amer. Math. Soc.* **359** (2007), 4511-4535.

[To1] H. Toda, p-primary components of homotopy groups. IV. Compositions and toric constructions, *Mem. Coll. Sci. Univ. Kyoto. Ser. A Math.* **32** (1959), 297-33.

[To2] H. Toda, On homotopy groups of S^3-bundles over spheres, *J. Math. Kyoto Univ.* **2** (1963), 193-207.

Department of Mathematics, University of Southampton, Southampton SO17 1BJ, United Kingdom

E-mail address: S.D.Theriault@soton.ac.uk